

Solution to the Mock-up Examination in the Course Category Theory

You have **120 minutes** at your disposal.

You may use one A4 page with handwritten notes. Any other written or electronic aids are not permitted except for normal watches. Carrying forbidden devices, even switched off, will be considered a cheating attempt.

Write your full name and matriculation number clearly legible on this cover sheet, as well as your name in the header on each sheet. Hand in all sheets. Leave them stapled together. Use only **pens** and **neither** the colour **red nor green**.

Check that you have received all the sheets. Questions can be found on **pages 1–15**. There are 5 questions for a total of 100 points. You may use the back of the sheets for secondary calculations. If you use the back of a sheet to answer, clearly mark what belongs to which question and indicate in the corresponding question where all parts of your answer can be found. Cross out everything that should not be graded.

With your signature, you confirm that you are in sufficiently good health at the beginning of the examination and that you accept this examination bindingly.

Last name (in CAPITAL LETTERS):

First name (in CAPITAL LETTERS):

Matriculation number:

Program of study:

Hierby I confirm the correctness of the above information:

Signature

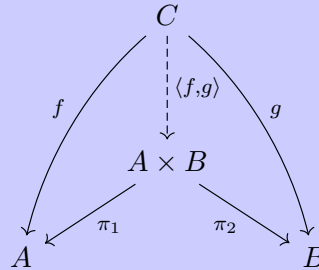
Please leave the following table blank:

Question	1	2	3	4	5	Σ
Points	15	20	15	30	20	100
Score						

Solution to Question 1 (Theorems and Definitions):**(15 points)**

- a) [5 points] Write down the definition for a category to have binary products. Give an example of a category with binary products.

PROPOSED SOLUTION: A category \mathcal{C} has binary products if for any two objects $A, B \in \mathcal{C}$ there is an object $A \times B$ and two morphisms $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ such that for any object $C \in \mathcal{C}$ and morphisms $f : C \rightarrow A$ and $g : C \rightarrow B$, there is a unique morphism $\langle f, g \rangle : C \rightarrow A \times B$ that makes the diagram



commute.

An example of category with binary products is the category of sets, for two sets A and B , the product is the set $\{(a, b) \mid a \in A, b \in B\}$ of pairs with components from A and B , respectively.

- 1 point for fixing two objects
- 1 point for the data that form the product, i.e., object $A \times B$ and the projection maps
- 2 points for the universal property
- 1 point for an example

- b) [5 points] Give an example of a category without neither product nor coproduct. Briefly justify your answer.

PROPOSED SOLUTION: The discrete category



with two objects A, B and no morphisms other than id_A and id_B has no products: There is no object that has arrows to both of the objects A and B .

Similarly, as there is no objects that has arrows from both A and B , this category does not have coproducts.

- 1 point for the construction
- 2 points for explaining why products do not exist
- 2 points for explaining why coproducts do not exist

c) [5 points] Write down the statement of Yoneda's lemma.

PROPOSED SOLUTION: Let \mathcal{C} be a locally small category. For any object $A \in \mathcal{C}$, let $yA \in \text{Set}^{\mathcal{C}^{\text{op}}}$ denote the presheaf represented by A . Let X be any presheaf on \mathcal{C} . Then the Hom-set

$$\text{Hom}_{\text{Set}^{\mathcal{C}^{\text{op}}}}(yA, X)$$

is isomorphic to

$$X(A)$$

naturally in A and X .

- 1 point for locally small
- 1 point for yA or $\text{Hom}_{\text{Set}^{\mathcal{C}^{\text{op}}}}(-, A)$
- 1 point for the isomorphism
- 1 point for naturality in A
- 1 point for naturality in X

Solution to Question 2 (Projectiveness):**(20 points)**

A notion related to the existence of choice functions is that of being *projective*: an object P is said to be projective if for any epi $e : E \rightarrow X$ and arrow $f : P \rightarrow X$ there is some (not necessarily unique) arrow $\bar{f} : P \rightarrow E$ such that $e \circ \bar{f} = f$, as indicated in the following diagram:

$$\begin{array}{ccc} & & E \\ & \nearrow \bar{f} & \downarrow e \\ P & \xrightarrow{f} & X \end{array}$$

- a) [5 points] The axiom of choice says: Let I be an index set, and for each $i \in I$, let A_i be a nonempty set. Then there exists a *choice function* $f : I \rightarrow \bigcup_{i \in I} A_i$, such that for each $i \in I$, we have $f(i) \in A_i \subseteq \bigcup_{i \in I} A_i$.

Use the axiom of choice to prove that for every surjection $e : E \rightarrow X$, there exists a function $s : X \rightarrow E$, called a *section* of e , such that $e \circ s = \text{id}_X$.

PROPOSED SOLUTION: Consider the family $(e^{-1}(\{x\}))_{x \in X}$, where $e^{-1}(U)$ denotes the preimage of $U \subseteq X$ under e . By surjectivity of e , each $e^{-1}(\{x\})$ contains at least one element. Thus, by the axiom of choice, there is a choice function s that for each $x \in X$ returns an (arbitrary) element of $e^{-1}(\{x\})$. Since $e^{-1}(\{x\}) \subseteq E$ for any x , s can be viewed as a function $s : X \rightarrow E$. For any $x \in X$ we have

$$\begin{aligned} s(x) &\in e^{-1}(\{x\}) && \text{(by } s \text{ being a choice function)} \\ \implies e(s(x)) &\in \{x\} && \text{(by def. of preimage)} \\ \implies e(s(x)) &= x. \end{aligned}$$

so s is indeed a section of e .

- 1 point for defining the family for applying the axiom of choice
- 1 point for arguing that family is nonempty
- 1 point for applying the axiom of choice to obtain the choice function
- 2 points for checking the choice function is a section

- b) [5 points] Assume the axiom of choice. Indicate whether there exists a non-projective object in the category **Set**. Briefly justify your answer.

PROPOSED SOLUTION: No, there is no non-projective object in **Set**.

First note that the epis in **Set** are exactly the surjections. From the axiom of choice it follows that, e has a *section* s , i.e., a morphism $X \rightarrow E$ such that $e \circ s = \text{id}_X$. Now let P be any set and $f : P \rightarrow X$ be any function. To prove that P is projective, define

$$\bar{f} := s \circ f$$

and verify

$$\begin{aligned} & e \circ \bar{f} \\ &= e \circ s \circ f \\ &= \text{id}_X \circ f \\ &= f. \end{aligned}$$

- 1 point for “no non-projective object”
- 1 point for realizing epis are surjections
- 1 point for the definition of \bar{f}
- 2 points for checking the equality

- c) [10 points] A retract of P is an object P' such that there exist morphisms $e' : P \rightarrow P'$ and $s : P' \rightarrow P$ such that $e' \circ s = \text{id}_{P'}$.

Show that in any category, the retract of a projective object is also projective. Include *both* of the following in your proof:

- a sequence of equations, *and*
- a commutative diagram with all the morphisms occurring in your equational proof.

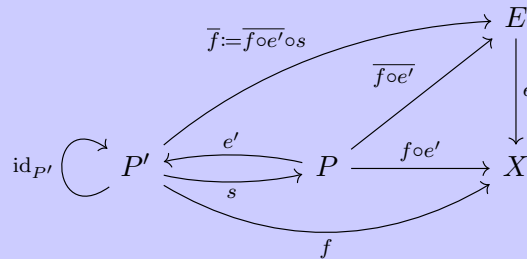
PROPOSED SOLUTION: To show that P' is projective, let $f : P' \rightarrow X$ be any morphism and define $\bar{f} : P' \rightarrow E$ as

$$\bar{f} := \overline{f \circ e'} \circ s$$

where $\overline{f \circ e'}$ is constructed using the projectivity of P and observe

$$\begin{aligned} e \circ \bar{f} &= e \circ \overline{f \circ e'} \circ s \\ &= f \circ e' \circ s \\ &= f \circ \text{id}_{P'} \\ &= f \end{aligned}$$

or, equivalently, that the following diagram



commutes.

- 1 point for fixing the $f : P' \rightarrow X$, this indicates you are checking the correct definition
- 3 points for defining the section
- 3 points for the equational proof
 - 1 point for using $e' \circ s = \text{id}_{P'}$
 - 2 points for a followable, readable equational proof
- 3 points for the commutative diagram (it's okay if $f \circ e'$ is missing)

Solution to Question 3 (Examples):**(15 points)**

- a) [8 points] Give an example of a finite limit in the category that consists of sets and binary relations. Specify the underlying diagram of the limit. Justify your answer. Make sure that your answer includes the explicit construction of the finite limit.

PROPOSED SOLUTION: The terminal object is the limit of the (finite) empty diagram. In the category of sets and binary relations, the terminal element is the empty set \emptyset . To show that \emptyset is a terminal object, let A be any set, and verify that there exists exactly one morphism from A to \emptyset :

Let R be any morphism $A \rightarrow \emptyset$. In the category of sets and binary relations, morphisms are binary relations, thus $R \subseteq A \times \emptyset = \emptyset$. The empty set has exactly one subset, namely the empty set, so $R = \emptyset$.

- 2 points for describe the diagram for the limit (it is sufficient to either define the functor or say it is a product/equalizer/terminal)
- 3 points for the construction
 - 1 point for constructing the object
 - 2 points for the projection arrows
- 3 points for proving the universal property

- b) [7 points] Give an example of a category with a zero object (i.e., the initial object and the terminal object coincide). Justify your answer.

PROPOSED SOLUTION: The initial object in the category of sets and binary relations is \emptyset , coinciding with the terminal object. The proof is the same as above but with $R \subseteq \emptyset \times A$.

- 1 point for naming the category
- 2 points for constructing the zero object
- 2 points for proving it is the initial object
- 2 points for proving it is the terminal object

Solution to Question 4 (Adjunction):**(30 points)**

a) [20 points] Consider the adjunction $F \dashv U$ such that $F : \mathbf{Set} \rightarrow \mathbf{Grp}$ is the free group functor and $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ is the forgetful functor.

(i) [10 points] Conclude that U preserves finite products by *both* of the following two approaches

1. Show that U is representable.

PROPOSED SOLUTION: By the definition of $F \dashv U$

$$\mathrm{Hom}_{\mathbf{Grp}}(F(A), B) \simeq \mathrm{Hom}_{\mathbf{Set}}(A, U(B))$$

naturally in A and B . Setting $A = \{*\}$, the right hand side $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, U(B))$ is naturally isomorphic to $U(B)$. Therefore

$$\mathrm{Hom}_{\mathbf{Grp}}(F(\{*\}), B) \simeq U(B)$$

naturally in B , which is precisely the statement that U is represented by $F(\{*\})$, the free group on one generator.

- 2 points for the isomorphism for adjunction
- 1 point for naturality
- 2 points for plugging in the A to be the one-point set

2. Conclude that U preserves finite products by using a general theorem about adjunctions. Write down the full statement you rely on.

PROPOSED SOLUTION: Right adjoints preserve limits. Let

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathcal{D} \\ & U & \end{array}$$

be an adjunction, J be a small category and $D : J \rightarrow \mathcal{D}$ be a diagram of shape J in \mathcal{D} and suppose the limit $\varprojlim D$ exists. Then $\varprojlim UD$ and is isomorphic to $U(\varprojlim D)$.

The product of $A_1, \dots, A_n \in \mathcal{D}$ is a limit, by taking J to be the discrete category on $\{1, \dots, n\}$ and $D(n) = A_n$. Therefore, right adjoints preserve finite products.

Taking $\mathcal{C} = \mathbf{Set}$, $\mathcal{D} = \mathbf{Grp}$ and F, U as above proves the claim.

- 2 points for “right adjoint preserve limits”
- 1 point for $\varprojlim UD \cong U \varprojlim D$
- 1 point for finite products are limits
- 1 point for plugging in $F \dashv U$

(ii) [10 points] Describe the unit and the counit of the adjunction $F \dashv U$.

PROPOSED SOLUTION: The unit $\eta : \text{id}_{\text{Set}} \Rightarrow UF$ is defined componentwise as $\eta_A(a) := a$, as a function $A \rightarrow UF(A)$. The codomain of this function is the underlying set of the free group $F(A)$ on the set A of generators. The first a is an element of the set A and the second a is the same element considered as a generator of $F(A)$.

To define the counit $\varepsilon : FU \Rightarrow \text{id}_{\text{Grp}}$ we define a group homomorphism $\varepsilon_B : FU(B) \rightarrow B$ for each $B \in \text{Grp}$. A group homomorphism from a free group $F(U(B))$ is fully determined by its action on the generators, i.e., the elements of the set $U(B)$. Elements $b \in U(B)$ are exactly elements of the group B , so we set $\varepsilon_B(b) = b$.

- 5 points for the unit
 - 1 point for specifying the domain and codomain
 - 1 point for pointing out the unit is a function
 - 3 points for the action of the function
- 5 points for the counit
 - 1 point for specifying the domain and codomain
 - 1 point for pointing out the counit is a group homomorphism
 - 3 points for the action of the function

- b) [10 points] Consider the diagonal functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Set} \times \mathbf{Set}$. Indicate whether Δ has a left adjoint and whether it has a right adjoint. Justify your answer.

PROPOSED SOLUTION: Δ has both a left and a right adjoint, given by the coproduct and product, respectively.

A left adjoint $L \vdash \Delta$ is a functor $L : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set}$ such that

$$\mathrm{Hom}_{\mathbf{Set}}(L(A, B), C) \simeq \mathrm{Hom}_{\mathbf{Set} \times \mathbf{Set}}((A, B), \Delta C),$$

naturally in $(A, B) \in \mathbf{Set} \times \mathbf{Set}$ and $C \in \mathbf{Set}$. The right hand side is naturally isomorphic to

$$\mathrm{Hom}_{\mathbf{Set}}(A, C) \times \mathrm{Hom}_{\mathbf{Set}}(B, C)$$

where we have used $\Delta C := (C, C)$.

In other words, L needs to be chosen such that a function $f : L(A, B) \rightarrow C$ is the same as a pair of functions $A \rightarrow C$ and $B \rightarrow C$. This is precisely the universal property of the disjoint union $A + B =: L(A, B)$.

Similarly, a right adjoint $\Delta \dashv R$ needs to satisfy

$$\mathrm{Hom}_{\mathbf{Set} \times \mathbf{Set}}(\Delta A, (B, C)) \simeq \mathrm{Hom}_{\mathbf{Set}}(A, R(B, C))$$

or equivalently

$$\mathrm{Hom}_{\mathbf{Set}}(A, B) \times \mathrm{Hom}_{\mathbf{Set}}(A, C) \simeq \mathrm{Hom}_{\mathbf{Set}}(A, R(B, C))$$

which is precisely the universal property of the product $B \times C =: R(B, C)$.

- 5 points for the left adjoint
 - 1 point for the left adjoint exists
 - 4 points for the construction of the left adjoint and a rigorous, readable proof
- 5 points for the right adjoint
 - 1 point for the right adjoint exists
 - 4 points for the construction of the right adjoint and a rigorous, readable proof

Solution to Question 5 (Power Set Monad):**(20 points)**

Consider the association that sends each set X to its power set $\mathcal{P}(X)$.

- a) [5 points] Prove that the association $X \mapsto \mathcal{P}(X)$ defines a functor $\mathcal{P} : \text{Set} \rightarrow \text{Set}$.

PROPOSED SOLUTION: To show that \mathcal{P} defines a functor, we define for each function $f : X \rightarrow Y$ a function $\mathcal{P}(f) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ as

$$\mathcal{P}(f)(U \subseteq X) := \{f(x) \mid x \in U\}$$

and check that for any X and $U \subseteq X$

$$\begin{aligned} & \mathcal{P}(\text{id}_X)(U) \\ &= \{\text{id}_X(x) \mid x \in U\} \\ &= \{x \mid x \in U\} \\ &= U \\ &= \text{id}_{\mathcal{P}(X)}(U) \end{aligned}$$

and that for any X, Y, Z , $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and $U \subseteq X$

$$\begin{aligned} & \mathcal{P}(f \circ g)(U) \\ &= \{f(g(x)) \mid x \in U\} \\ &= \{f(y) \mid y = g(x) \text{ and } x \in U\} \\ &= \{f(y) \mid y \in \{g(x) \mid x \in U\}\} \\ &= \{f(y) \mid y \in \mathcal{P}(g)(U)\} \\ &= \mathcal{P}(f)(\mathcal{P}(g)(U)) \end{aligned}$$

- 2 points for defining $\mathcal{P}(f)$
- 1 point for unfolding the definition of $\mathcal{P}(\text{id}_X)$ in the identity law proof, and show $\mathcal{P}(\text{id}_X)(U) = U$ or $\mathcal{P}(\text{id}_X)(U) = \text{id}_{\mathcal{P}(X)}(U)$
- 2 points for the proof that \mathcal{P} respects composition

- b) [7 points] Specify a monad structure on the endofunctor \mathcal{P} by defining the unit η and the multiplication μ . You do not need to prove the monad laws, nor prove the naturality of η and μ .

PROPOSED SOLUTION: A monad structure on an endofunctor \mathcal{P} consists of two natural transformations $\eta : \text{id}_{\text{Set}} \Rightarrow \mathcal{P}$ and $\mu : \mathcal{P} \circ \mathcal{P} \Rightarrow \mathcal{P}$. The unit

$$\begin{aligned}\eta_X &: X \rightarrow \mathcal{P}(X) \\ \eta_X(x) &:= \{x\}\end{aligned}$$

of the monad “injects” a value x into the powerset as $\{x\}$. The multiplication

$$\begin{aligned}\mu_X &: \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(X) \\ \mu_X(\mathcal{U}) &:= \bigcup_{U \in \mathcal{U}} U\end{aligned}$$

of the monad takes a set \mathcal{U} of subsets U of X and returns the union of the U s, yielding a subset of X .

- 1 point for giving the correct type of η or of η_X
- 1 point for giving the correct type of μ or of μ_X
- 2 points for the definition of unit
- 3 points for the definition of multiplication

- c) [8 points] The associativity law for a monad \mathcal{P} can be stated as the condition that the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{P}\mathcal{P}\mathcal{P} & \xrightarrow{\mathcal{P}\mu} & \mathcal{P}\mathcal{P} \\
 \mu_{\mathcal{P}} \downarrow & & \downarrow \mu \\
 \mathcal{P}\mathcal{P} & \xrightarrow{\mu} & \mathcal{P}
 \end{array}$$

Describe the two natural transformations $\mathcal{P}\mu$ and $\mu_{\mathcal{P}}$ componentwise.

PROPOSED SOLUTION: For any set $X \in \text{Set}$ and $\mathfrak{U} \in \mathcal{P}\mathcal{P}\mathcal{P}(X)$

$$\begin{aligned}
 & (\mathcal{P}\mu)_X(\mathfrak{U}) \\
 &= \mathcal{P}(\mu_X)(\mathfrak{U}) \\
 &= \{\mu_X(\mathcal{U}) \mid \mathcal{U} \in \mathfrak{U}\} \\
 &= \left\{ \bigcup_{U \in \mathcal{U}} U \mid \mathcal{U} \in \mathfrak{U} \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 & (\mu_{\mathcal{P}})_X(\mathfrak{U}) \\
 &= \mu_{\mathcal{P}(X)}(\mathfrak{U}) \\
 &= \bigcup_{\mathcal{U} \in \mathfrak{U}} \mathcal{U}.
 \end{aligned}$$

- 4 points for describing $\mathcal{P}\mu$
- 4 points for describing $\mu_{\mathcal{P}}$

- d) [5 points] Give an example of an algebra of the monad (\mathcal{P}, η, μ) . (*Hint*: Consider the supremum operation on a finite lattice, e.g., the max operation on a finite set of natural numbers.)

PROPOSED SOLUTION: Any sup-lattice L forms an \mathcal{P} -algebra $\mathcal{P}(L) \xrightarrow{\vee} L$. For example, the \vee operation on subsets of the Boolean lattice $\mathbb{B} = \{0 \leq 1\}$

$$\begin{aligned} \vee & : \mathcal{P}(\mathbb{B}) \rightarrow \mathbb{B} \\ \vee \emptyset & := 0 \\ \vee \{b_1, \dots, b_n\} & := b_1 \vee \dots \vee b_n \text{ for } n \in \{1, \dots\} \end{aligned}$$

is a \mathcal{P} -algebra with carrier \mathbb{B} . The diagram

$$\begin{array}{ccc} \mathcal{P}\mathcal{P}\mathbb{B} & \xrightarrow{\mathcal{P}(\vee)} & \mathcal{P}\mathbb{B} \\ \mu \downarrow & & \downarrow \vee \\ \mathcal{P}\mathbb{B} & \xrightarrow{\vee} & \mathbb{B} \end{array}$$

commutes because for any $\mathcal{U} \in \mathcal{P}\mathcal{P}(\mathbb{B})$,

$$\begin{aligned} & \vee (\mathcal{P}(\vee)(\mathcal{U})) \\ &= \vee (\{\vee U \mid U \in \mathcal{U}\}) \\ &= \vee \left(\bigcup_{U \in \mathcal{U}} U \right) \\ &= \vee \mu_X(\mathcal{U}) \end{aligned}$$

where the second equality follows from the associativity of \vee .

- 2 points for specifying the definition of monad algebra
- 3 points for the concrete example
 - 1 point for the underlying set of the algebra
 - 2 points for the evaluation operation