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Mock-up Examination in the Course
Category Theory

You have **120 minutes** at your disposal.

You may use one A4 page with handwritten notes. Any other written or electronic aids are not permitted except for normal watches. Carrying forbidden devices, even switched off, will be considered a cheating attempt.

Write your full name and matriculation number clearly legible on this cover sheet, as well as your name in the header on each sheet. Hand in all sheets. Leave them stapled together. Use only **pens** and **neither** the colour **red nor green**.

Check that you have received all the sheets. Questions can be found on **pages 1–14**. There are 5 questions for a total of 100 points. You may use the back of the sheets for secondary calculations. If you use the back of a sheet to answer, clearly mark what belongs to which question and indicate in the corresponding question where all parts of your answer can be found. Cross out everything that should not be graded.

With your signature, you confirm that you are in sufficiently good health at the beginning of the examination and that you accept this examination bindingly.

Last name (in CAPITAL LETTERS):

First name (in CAPITAL LETTERS):

Matriculation number:

Program of study:

Hierby I confirm the correctness of the above information: ______________________
Signature

Please leave the following table blank:

Question	1	2	3	4	5	$\sum$
Points	15	20	15	30	20	100
Score						

Question 1 (Theorems and Definitions): (15 points)

a) [5 points] Write down the definition for a category to have binary products. Give an example of a category with binary products.

b) [5 points] Give an example of a category without neither product nor coproduct. Briefly justify your answer.

c) [5 points] Write down the statement of Yoneda's lemma.

Question 2 (Projectiveness): **(20 points)**

A notion related to the existence of choice functions is that of being *projective*: an object P is said to be projective if for any epi $e : E \to X$ and arrow $f : P \to X$ there is some (not necessarily unique) arrow $\overline{f} : P \to E$ such that $e \circ \overline{f} = f$, as indicated in the following diagram:

$$\begin{tikzcd} & E \arrow[d, "e"] \\ P \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

a) [5 points] The axiom of choice says: Let I be an index set, and for each $i \in I$, let A_i be a nonempty set. Then there exists a *choice function* $f : I \to \bigcup_{i \in I} A_i$, such that for each $i \in I$, we have $f(i) \in A_i \subseteq \bigcup_{i \in I} A_i$.

Use the axiom of choice to prove that for every surjection $e : E \to X$, there exists a function $s : X \to E$, called a *section* of e, such that $e \circ s = \mathrm{id}_X$.

b) [5 points] Assume the axiom of choice. Indicate whether there exists a non-projective object in the category Set. Briefly justify your answer.

c) [10 points] A retract of P is an object P' such that there exist morphisms $e' : P \to P'$ and $s : P \to P'$ such that $e' \circ s = \mathrm{id}_{P'}$.

Show that in any category, the retract of a projective object is also projective. Include *both* of the following in your proof:

- a sequence of equations, *and*
- a commutative diagram with all the morphisms occurring in your equational proof.

Question 3 (Examples): **(15 points)**

a) [8 points] Give an example of a finite limit in the category that consists of sets and binary relations. Specify the underlying diagram of the limit. Justify your answer. Make sure that your answer includes the explicit construction of the finite limit.

b) [7 points] Give an example of a category with a zero object (i.e., the initial object and the terminal object coincide). Justify your answer.

Question 4 (Adjunction): **(30 points)**

a) [20 points] Consider the adjunction $F \dashv U$ such that $F : \mathsf{Set} \to \mathsf{Grp}$ is the free group functor and $U : \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor.

(i) [10 points] Conclude that U preserves finite products by *both* of the following two approaches

1. Show that U is representable.

2. Conclude that U preserves finite products by using a general theorem about adjunctions. Write down the full statement you rely on.

(ii) [10 points] Describe the unit and the counit of the adjunction $F \dashv U$.

b) [10 points] Consider the diagonal functor $\Delta : \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$. Indicate whether Δ has a left adjoint and whether it has a right adjoint. Justify your answer.

Question 5 (Power Set Monad): **(20 points)**

Consider the association that sends each set X to its power set $\mathcal{P}(X)$.

a) [5 points] Prove that the association $X \mapsto \mathcal{P}(X)$ defines a functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$.

b) [7 points] Specify a monad structure on the endofunctor $\mathcal{P}$ by defining the unit η and the multiplication μ. You do not need to prove the monad laws, nor prove the naturality of η and μ.

c) [8 points] The associativity law for a monad $\mathcal{P}$ can be stated as the condition that the following diagram commutes.

$$\begin{tikzcd} \mathcal{P}\mathcal{P}\mathcal{P} \arrow[r, "\mathcal{P}\mu"] \arrow[d, "\mu_{\mathcal{P}}"'] & \mathcal{P}\mathcal{P} \arrow[d, "\mu"] \\ \mathcal{P}\mathcal{P} \arrow[r, "\mu"'] & \mathcal{P} \end{tikzcd}$$

Describe the two natural transformations $\mathcal{P}\mu$ and $\mu_{\mathcal{P}}$ componentwise.

d) [5 points] Give an example of an algebra of the monad $(\mathcal{P}, \eta, \mu)$. (*Hint*: Consider the supremum operation on a finite lattice, e.g., the max operation on a finite set of natural numbers.)