Automated Theorem Proving

Lecture 8: Semantic Tableaux

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3.17 Semantic Tableaux

Literature:

M. Fitting: *First-Order Logic and Automated Theorem Proving*, Springer-Verlag, New York, 1996, chapters 3, 6, 7.

R. M. Smullyan: *First-Order Logic*, Dover Publ., New York, 1968, revised 1995.

Like resolution, semantic tableaux were developed in the 1960s, independently by Zbigniew Lis and Raymond Smullyan on the basis of work by Gentzen in the 1930s and of Beth in the 1950s.

Idea

Idea (for the propositional case):

A set $\{F \wedge G\} \cup N$ of formulas has a model if and only if $\{F \wedge G,\ F,\ G\} \cup N$ has a model.

A set $\{F \vee G\} \cup N$ of formulas has a model if and only if $\{F \vee G,\ F\} \cup N$ or $\{F \vee G,\ G\} \cup N$ has a model.

(And similarly for other connectives.)

To avoid duplication, represent sets as paths of a tree.

Continue splitting until two complementary formulas are found $\Rightarrow$ inconsistency detected.

A Tableau for $\{P \wedge \neg(Q \vee \neg R),\ \neg Q \vee \neg R\}$

1. $P \wedge \neg(Q \vee \neg R)$
2. $\neg Q \vee \neg R$

3.	$\neg Q$	4.	$\neg R$
5.	P	10.	P
6.	$\neg(Q \vee \neg R)$	11.	$\neg(Q \vee \neg R)$
7.	$\neg Q$		
8.	$\neg\neg R$		
9.	R		

This tableau is not "maximal"; however, the first "path" is. This path is not "closed"; hence the set $\{1, 2\}$ is satisfiable. (These notions will all be defined below.)

Properties

Properties of tableau calculi:

analytic: inferences correspond closely to the logical meaning of the symbols.

goal-oriented: inferences operate directly on the goal to be proved.

global: some inferences affect the entire proof state (set of formulas), as we will see later.

Propositional Expansion Rules

Expansion rules are applied to the formulas in a tableau and expand the tableau at a leaf. We append the conclusions of a rule (horizontally or vertically) at a *leaf* whenever the premise of the expansion rule matches a formula appearing *anywhere* on the path from the root to that leaf.

Negation Elimination

$$\frac{\neg\neg F}{F} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top}$$

Propositional Expansion Rules

α-Expansion

(for formulas that are essentially conjunctions: append subformulas α_1 and α_2 one on top of the other)

$$\frac{\alpha}{\begin{array}{c}\alpha_1 \\ \alpha_2\end{array}}$$

β-Expansion

(for formulas that are essentially disjunctions:
append β_1 and β_2 horizontally, i.e., branch into β_1 and β_2)

$$\frac{\beta}{\beta_1 \mid \beta_2}$$

Classification of Formulas

conjunctive			disjunctive		
α	α_1	α_2	β	β_1	β_2
$F \wedge G$	F	G	$\neg(F \wedge G)$	$\neg F$	$\neg G$
$\neg(F \vee G)$	$\neg F$	$\neg G$	$F \vee G$	F	G
$\neg(F \rightarrow G)$	F	$\neg G$	$F \rightarrow G$	$\neg F$	G

We assume that the binary connective $\leftrightarrow$ has been eliminated in advance.

Tableaux: Notions

A semantic tableau is a marked (by formulas), finite, unordered tree and inductively defined as follows: Let $\{F_1, \ldots, F_n\}$ be a set of formulas.

(i) The tree consisting of a single path

$$\begin{array}{c} F_1 \\ \vdots \\ F_n \end{array}$$

is a tableau for $\{F_1, \ldots, F_n\}$.
(We do not draw edges if nodes have only one successor.)

Tableaux: Notions

(ii) If T is a tableau for $\{F_1, \dots, F_n\}$ and if T' results from T by applying an expansion rule, then T' is also a tableau for $\{F_1, \dots, F_n\}$.

Note: We may also consider the *limit tableau* of a tableau expansion; this can be an *infinite* tree.

Tableaux: Notions

A path (from the root to a leaf) in a tableau is called closed if it either contains $\perp$ or else it contains both some formula F and its negation $\neg F$. Otherwise the path is called open.

A tableau is called closed if all paths are closed.

A tableau proof for F is a closed tableau for $\{\neg F\}$.

Tableaux: Notions

A path π in a tableau is called maximal if for each formula F on π that is neither a literal nor $\bot$ nor $\top$ there exists a node in π at which the expansion rule for F has been applied.

In that case, if F is a formula on π, π also contains:

(i) α_1 and α_2 if F is a α-formula,

(ii) β_1 or β_2 if F is a β-formula, and

(iii) F' if F is a negation formula, and F' the conclusion of the corresponding elimination rule.

A tableau is called maximal if each path is closed or maximal.

Tableaux: Notions

A tableau is called strict if for each formula the corresponding expansion rule has been applied at most once on each path containing that formula.

An Example Proof

One starts out from the negation of the formula to be proved.

1. $\neg((P \to (Q \to R)) \to ((P \lor S) \to ((Q \to R) \lor S)))$
2. $(P \to (Q \to R))$ $[1_1]$
3. $\neg((P \lor S) \to ((Q \to R) \lor S))$ $[1_2]$
4. $P \lor S$ $[3_1]$
5. $\neg((Q \to R) \lor S))$ $[3_2]$
6. $\neg(Q \to R)$ $[5_1]$
7. $\neg S$ $[5_2]$

8. $\neg P$ $[2_1]$ 9. $Q \to R$ $[2_2]$

10. P $[4_1]$ 11. S $[4_2]$

There are three paths, each of them closed.

Properties of Propositional Tableaux

We assume that T is a tableau for $\{F_1, \ldots, F_n\}$.

Theorem 3.17.1:
$\{F_1, \ldots, F_n\}$ satisfiable $\Leftrightarrow$ some path (i.e., the set of its formulas) in T is satisfiable.

Proof:
($\Leftarrow$) Trivial, since every path contains in particular $F_1, \ldots, F_n$.
($\Rightarrow$) By induction over the structure of T. $\square$

Corollary 3.17.2:
T closed $\Rightarrow$ $\{F_1, \ldots, F_n\}$ unsatisfiable

Properties of Propositional Tableaux

Theorem 3.17.3:
Every strict propositional tableau expansion is finite.

Proof:
New formulas resulting from expansion are $\bot$, $\top$, or subformulas of the expanded formula (modulo de Morgan's law), so the number of formulas that can occur is finite. By strictness, on each path a formula can be expanded at most once. Therefore, each path is finite, and a finitely branching tree with finite paths is finite by König's lemma. $\square$

Conclusion: Strict maximal tableaux can be effectively constructed.

Refutational Completeness

A set $\mathcal{H}$ of propositional formulas is called a Hintikka set if

(1) there is no $P \in \Pi$ with $P \in \mathcal{H}$ and $\neg P \in \mathcal{H}$;

(2) $\bot \notin \mathcal{H}$, $\neg\top \notin \mathcal{H}$;

(3) if $\neg\neg F \in \mathcal{H}$, then $F \in \mathcal{H}$;

(4) if $\alpha \in \mathcal{H}$, then $\alpha_1 \in \mathcal{H}$ and $\alpha_2 \in \mathcal{H}$;

(5) if $\beta \in \mathcal{H}$, then $\beta_1 \in \mathcal{H}$ or $\beta_2 \in \mathcal{H}$.

Refutational Completeness

Lemma 3.17.4 (Hintikka's Lemma):
Every Hintikka set is satisfiable.

Proof:
Let $\mathcal{H}$ be a Hintikka set. Define a valuation $\mathcal{A}$ by $\mathcal{A}(P) = 1$ if $P \in \mathcal{H}$ and $\mathcal{A}(P) = 0$ otherwise. Then show that $\mathcal{A}(F) = 1$ for all $F \in \mathcal{H}$ by induction over the size of formulas. $\square$

Refutational Completeness

Theorem 3.17.5:
Let π be a maximal open path in a tableau. Then the set of formulas on π is satisfiable.

Proof:
We show that set of formulas on π is a Hintikka set: Conditions (3), (4), (5) follow from the fact that π is maximal; conditions (1) and (2) follow from the fact that π is open and from maximality for the second negation elimination rule. $\square$

Refutational Completeness

Theorem 3.17.6:
$\{F_1, \ldots, F_n\}$ satisfiable $\Leftrightarrow$ there exists no closed strict tableau for $\{F_1, \ldots, F_n\}$.

Proof:
($\Rightarrow$) Clear by Cor. 3.17.2.
($\Leftarrow$) Let T be a strict maximal tableau for $\{F_1, \ldots, F_n\}$ and let π be an open path in T. By the previous theorem, the set of formulas on π is satisfiable, and hence by Theorem 3.17.1 the set $\{F_1, \ldots, F_n\}$, is satisfiable.

$\square$

Consequences

The validity of a propositional formula F can be established by constructing a strict maximal tableau for $\{\neg F\}$:

- T closed $\Leftrightarrow$ F valid.
- It suffices to test complementarity of paths w.r.t. atomic formulas (cf. reasoning in the proof of Theorem 3.17.5).
- Which of the potentially many strict maximal tableaux one computes does not matter. In other words, tableau expansion rules can be applied don't-care nondeterministically ("proof confluence").
- The expansion strategy, however, can have a dramatic impact on the tableau size.

A Variant of the β-Rule

Since $F \vee G \models\!\!\!\!\!= F \vee (G \wedge \neg F)$, the β expansion rule

$$\frac{\beta}{\beta_1 \mid \beta_2}$$

can be replaced by the following variant:

$$\frac{\beta}{\begin{array}{l|l} \beta_1 & \beta_2 \\ & \neg\beta_1 \end{array}}$$

A Variant of the β-Rule

The variant β-rule can lead to much shorter proofs, but it is not always beneficial.

In general, it is most helpful if $\neg\beta_1$ can be at most (iteratively) α-expanded.

3.18 Semantic Tableaux for First-Order Logic

There are two ways to extend the tableau calculus to quantified formulas:

- using ground instantiation,
- using free variables.

Tableaux with Ground Instantiation

Classification of quantified formulas:

universal		existential	
γ	$\gamma(t)$	δ	$\delta(t)$
$\forall x F$	$F\{x \mapsto t\}$	$\exists x F$	$F\{x \mapsto t\}$
$\neg \exists x F$	$\neg F\{x \mapsto t\}$	$\neg \forall x F$	$\neg F\{x \mapsto t\}$

Tableaux with Ground Instantiation

Idea:

Replace universally quantified formulas by appropriate ground instances.

γ-**expansion**

$$\frac{\gamma}{\gamma(t)} \quad \text{where } t \text{ is some ground term}$$

δ-**expansion**

$$\frac{\delta}{\delta(c)} \quad \text{where } c \text{ is a new Skolem constant}$$

Tableaux with Ground Instantiation

Skolemization becomes part of the calculus and needs not necessarily be applied in a preprocessing step. Of course, one could do Skolemization beforehand, and then the δ-rule would not be needed.

Note:

Skolem *constants* are sufficient:
In a δ-formula $\exists x\, F$, $\exists$ is the outermost quantifier
and x is the only free variable in F.

Tableaux with Ground Instantiation

Problems:

Having to guess ground terms is impractical.

Even worse, we may have to guess *several* ground instances, since strictness for γ is incomplete. For instance, constructing a closed tableau for

$$\{\forall x\, (P(x) \rightarrow P(f(x))),\ P(b),\ \neg P(f(f(b)))\}$$

is impossible without applying γ-expansion twice on one path.

Free-Variable Tableaux

An alternative approach:

Delay the instantiation of universally quantified variables.

Replace universally quantified variables by new free variables.

Intuitively, the free variables are universally quantified *outside* of the entire tableau.

Free-Variable Tableaux

γ-expansion

$$\frac{\gamma}{\gamma(x)} \quad \text{where } x \text{ is a new free variable}$$

δ-expansion

$$\frac{\delta}{\delta(f(x_1, \ldots, x_n))}$$

where f is a new Skolem function, and the x_i are the free variables in δ

Free-Variable Tableaux

Application of expansion rules must be supplemented by a substitution rule:

(iii) If T is a tableau for $\{F_1, \ldots, F_n\}$ and if σ is a substitution, then $T\sigma$ is also a tableau for $\{F_1, \ldots, F_n\}$.

The substitution rule may, potentially, modify all the formulas of a tableau. This feature is what makes the tableau method a *global proof method*.

Free-Variable Tableaux

One can show that it is sufficient to consider substitutions σ for which there is a path in T containing two *literals* $\neg A$ and B such that $\sigma = \text{mgu}(A, B)$. Such tableaux are called AMGU-Tableaux.

Example of a Free-Variable Tableau

1.	$\neg(\exists w \forall x\ P(x, w, f(x, w)) \rightarrow \exists w \forall x \exists y\ P(x, w, y))$	
2.	$\exists w \forall x\ P(x, w, f(x, w))$	1_1 $[\alpha]$
3.	$\neg \exists w \forall x \exists y\ P(x, w, y)$	1_2 $[\alpha]$
4.	$\forall x\ P(x, c, f(x, c))$	$2(c)$ $[\delta]$
5.	$\neg \forall x \exists y\ P(x, v_1, y)$	$3(v_1)$ $[\gamma]$
6.	$\neg \exists y\ P(b(v_1), v_1, y)$	$5(b(v_1))$ $[\delta]$
7.	$P(v_2, c, f(v_2, c))$	$4(v_2)$ $[\gamma]$
8.	$\neg P(b(v_1), v_1, v_3)$	$6(v_3)$ $[\gamma]$

7 and 8 are complementary (modulo unification):

$$\{v_2 \doteq b(v_1),\ c \doteq v_1,\ f(v_2, c) \doteq v_3\}$$

is solvable with an mgu $\sigma = \{v_1 \mapsto c,\ v_2 \mapsto b(c),\ v_3 \mapsto f(b(c), c)\}$, and hence, $T\sigma$ is a closed (linear) tableau for the formula in 1.

Free-Variable Tableaux

Problem:

Strictness for γ is still incomplete.
For instance, constructing a closed tableau for

$$\{\forall x\,(P(x) \rightarrow P(f(x))),\ P(b),\ \neg P(f(f(b)))\}$$

is impossible without applying γ-expansion twice on one path.

Semantic Tableaux vs. Resolution

- Tableaux: global, goal-oriented, “backward.”
- Resolution: local, “forward.”
- Goal-orientation can be advantageous if only a small subset of a large set of formulas is necessary for a proof.
  (Resolution provers saturate also those parts of the clause set that are irrelevant for proving the goal.)

Semantic Tableaux vs. Resolution

- Resolution can be combined with more powerful redundancy elimination methods; this is more difficult for the tableau method.
- Resolution can be refined to work well with equality; for tableaux this seems to be impossible.
- On the other hand, tableau calculi can be easily extended to other logics; in particular tableau provers are very successful in modal and description logics.