

Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 7: Resolution Continued

Exercise 7.1: Find a strict total ordering \succ on the ground atoms $P(b), P(c), Q, R$ such that

$$P(b) \vee \neg P(c) \succ_c \neg P(b) \vee P(c) \quad (1)$$

$$P(b) \vee P(b) \vee P(b) \vee R \succ_c P(b) \vee R \vee R \quad (2)$$

$$\neg P(b) \vee Q \succ_c P(c) \vee R \quad (3)$$

Exercise 7.2: Consider the following formulas:

$$F_1 = \forall x (S(x) \rightarrow \exists y (R(x, y) \wedge P(y)))$$

$$F_2 = \forall x (P(x) \rightarrow Q(x))$$

$$F_3 = \exists x S(x)$$

$$G = \exists x \exists y (R(x, y) \wedge Q(y))$$

Use ordered resolution to prove that $\{F_1, F_2, F_3\} \models G$. You may choose the selection function and the ordering on atoms.

Hint: You will need some preprocessing.

Exercise 7.3: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/2, Q/1, R/2\}$. Suppose that the atom ordering \succ compares ground atoms by comparing lexicographically first the predicate symbols ($P \succ Q \succ R$), then the size of the first argument, then the size of the second argument (if present). If at least one of the two atoms to be compared is nonground, \succ compares only the predicate symbols.

Let N be the following set of clauses:

$$P(f(x), x) \vee R(b, b) \quad (1)$$

$$\neg P(b, x) \vee \neg P(x, b) \vee Q(x) \quad (2)$$

$$Q(f(b)) \vee \neg Q(b) \vee R(f(x), b) \quad (3)$$

$$Q(b) \vee \neg R(f(x), f(x)) \quad (4)$$

$$\neg Q(x) \vee R(x, x) \quad (5)$$

- (a) Which literals are strictly maximal in the clauses of N ?
- (b) Which literals are maximal in the clauses of N ?
- (c) Which $Res_{sel}^>$ -inferences are possible if sel selects no literals? What are their conclusions?
- (d) Is there a $Res_{sel}^>$ -inference between the clause

$$P(x, f(x)) \vee R(b, b) \quad (1')$$

and clause (2) if sel selects no literals? Justify your answer.

- (e) Define a selection function sel such that N is saturated under $Res_{sel}^>$.

Exercise 7.4: In Sect. ?? of the lecture notes, the inference rules for ground resolution with ordering restrictions (without selection functions) are given by

(Ground) Ordered Resolution:

$$\frac{D \vee A \quad C \vee \neg A}{D \vee C} \quad \text{if } A \succ L \text{ for all } L \text{ in } D \text{ and } \neg A \succeq L \text{ for all } L \text{ in } C.$$

(Ground) Ordered Factorization:

$$\frac{C \vee A \vee A}{C \vee A} \quad \text{if } A \succeq L \text{ for all } L \text{ in } C.$$

This calculus is sound and refutationally complete for sets of ground clauses.

Suppose that we replace the ordering restriction for the first inference rule by “if $A \succ L$ for all L in D and $A \succeq L$ for all L in C .”

- (a) Is the calculus with this modification still sound? If yes, give a short explanation; if no, give a counterexample.
- (b) Is the calculus with this modification still refutationally complete? If yes, give a short explanation; if no, give a counterexample.

Exercise 7.5: Determine all strict total orderings \succ on the atomic formulas P, Q, R, S such that the associated clause ordering \succ_C satisfies the properties (1)–(3) simultaneously:

$$P \vee Q \succ_C \neg Q \quad (1)$$

$$R \vee Q \succ_C \neg P \vee \neg P \quad (2)$$

$$\neg R \vee \neg R \succ_C S \quad (3)$$

Exercise 7.6: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1, Q/1\}$. Suppose that the atom ordering \succ compares ground atoms by comparing lexicographically first the size of the argument and then the predicate symbols ($P \succ Q$). Let N be the following set of clauses:

$$\neg P(x) \vee P(f(x)) \quad (1)$$

$$\neg Q(f(b)) \vee P(f^3(b)) \quad (2)$$

$$Q(b) \vee Q(f(b)) \quad (3)$$

where $f^0(b)$ stands for b and $f^{i+1}(b)$ stands for $f(f^i(b))$.

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered w.r.t. the clause ordering \succ_C ?

(b) Construct the candidate interpretation $I_{G_\Sigma(N)}^\succ$ of the set of all ground instances of clauses in N . Is it a model of $G_\Sigma(N)$?

Exercise 7.7: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1, Q/1\}$. Suppose that the atom ordering \succ compares ground atoms by comparing lexicographically first the predicate symbols ($P \succ Q$) and then the size of the argument. Let N be the following set of clauses:

$$\neg Q(y) \vee P(y) \quad (1)$$

$$Q(x) \vee Q(f(x)) \quad (2)$$

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered w.r.t. the clause ordering \succ_C ?

(b) Construct the candidate interpretation $I_{G_\Sigma(N)}^\succ$ of the set of all ground instances of clauses in N .

Exercise 7.8: Let N be a set of ground clauses, and let \succ be a total and well-founded atom ordering. Prove or refute: If every clause in N is redundant w.r.t. N , then every clause in N is a tautology.

Exercise 7.9 (*): Give an example of two different first-order clauses F and G such that F entails G and G is not redundant w.r.t. $\{F\}$. If necessary, specify the atom ordering used by the redundancy criterion.

Exercise 7.10 (*): Give a clause C such that an “Ordered Resolution with Selection” inference is possible from C and C and the inference is not redundant w.r.t. $\{C\}$.