# Automated Theorem Proving

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## Exercises 10: Termination

**Exercise 10.1:** Let $\Sigma = (\{f/1,\ g/2,\ h/1,\ b/0,\ c/0\}, \emptyset)$ and let

$$\begin{aligned} t_1 &= g(h(x), h(c)), \\ t_2 &= g(x, x), \\ t_3 &= g(b, f(x)), \\ t_4 &= f(g(x, y)), \\ t_5 &= h(g(x, c)). \end{aligned}$$

Determine for each $1 \le i < j \le 5$ whether $t_i$ and $t_j$ are uncomparable or comparable (and if so, which term is larger) with respect to

(a) the lexicographic path ordering with the precedence $f \succ g \succ h \succ b \succ c$,

(b) the Knuth–Bendix ordering with the precedence $h \succ f \succ g \succ b \succ c$, where $h$ has weight 0, $b$ has weight 3, and all other symbols and variables have weight 1,

(c) the polynomial ordering over $\{n \in \mathbb{N} \mid n \ge 1\}$ with $P_f(X_1) = X_1 + 1$, $P_g(X_1, X_2) = 2X_1 + X_2 + 1$, $P_h(X_1) = 3X_1$, $P_b = 1$ and $P_c = 3$.

**Proposed solution.** (a)
$t_1 \succ t_2$
$t_1 \prec t_3$
$t_1 \prec t_4$
$t_1 \succ t_5$
$t_2$ and $t_3$ are uncomparable
$t_2 \prec t_4$
$t_2$ and $t_5$ are uncomparable
$t_3 \prec t_4$

$t_3$ and $t_5$ are uncomparable
$t_4 \succ t_5$

(b)
$t_1$ and $t_2$ are uncomparable
$t_1 \prec t_3$
$t_1 \prec t_4$
$t_1 \prec t_5$
$t_2$ and $t_3$ are uncomparable
$t_2$ and $t_4$ are uncomparable
$t_2$ and $t_5$ are uncomparable
$t_3$ and $t_4$ are uncomparable
$t_3 \succ t_5$
$t_4 \succ t_5$

(c)
$P_{t_1}(X) = 6X + 10$
$P_{t_2}(X) = 3X + 1$
$P_{t_3}(X) = X + 4$
$P_{t_4}(X, Y) = 2X + Y + 2$
$P_{t_5}(X) = 6X + 12$

$t_1 \succ t_2$
$t_1 \succ t_3$
$t_1$ and $t_4$ are uncomparable
$t_1 \prec t_5$
$t_2$ and $t_3$ are uncomparable
$t_2$ and $t_4$ are uncomparable
$t_2 \prec t_5$
$t_3 \prec t_4$
$t_3 \prec t_5$
$t_4$ and $t_5$ are uncomparable

**Exercise 10.2:** Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering on $\Omega$, and let $s, t \in \mathrm{T}_\Sigma(X)$.

(a) Prove: If $s$ contains a subterm $s' = f(s_1, \ldots, s_n)$ such that $\mathrm{var}(s') \supseteq \mathrm{var}(t)$ and $f \succ g$ for all function symbols $g$ occurring in $t$, then $s \succ_{\mathrm{lpo}} t$.

(b) Refute: If $s$ contains a subterm $s' = f(s_1, \ldots, s_n)$ such that $\mathrm{var}(s) \supseteq \mathrm{var}(t)$ and $f \succ g$ for all function symbols $g$ occurring in $t$, then $s \succ_{\mathrm{lpo}} t$.

**Proposed solution.** (a) Since every lpo is a simplification ordering, we know that $s \succeq_{\text{lpo}} s'$. Therefore it suffices to show that $s' \succ_{\text{lpo}} t$. We prove this by induction over the structure of $t$:

If $t$ is a variable, then by assumption $t \in \text{var}(t) \subseteq \text{var}(s')$; so $s' \succ_{\text{lpo}} t$ by case (1) of the lpo definition.

Otherwise $t = g(t_1, \dots, t_m)$ with $f \succ g$. By induction, $s' \succ_{\text{lpo}} t_j$ for all $1 \leq j \leq m$. So $s' \succ_{\text{lpo}} t$ by case (2)(b) of the lpo definition.

(b) Choose the precedence $f \succ g \succ h$, then $s = h(f(x), y) \not\succ_{\text{lpo}} g(y) = t$, but $f \succ g$ and $\text{var}(s) \supseteq \text{var}(t)$.

**Exercise 10.3:** Determine for each of the following statements whether they are true or false:

(1) $f(g(x)) \succ f(x)$ in every simplification ordering $\succ$.

(2) $f(f(x)) \succ f(y)$ in every simplification ordering $\succ$.

(3) If $\succ$ is an lpo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.

(4) If $\succ$ is a kbo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.

(5) If $\succ$ is an lpo, then $h(f(x), y, y) \succ h(x, z, z)$.

(6) If $\succ$ is a kbo, then $h(f(x), f(y), z) \succ h(x, f(z), y)$.

(7) There is a reduction ordering $\succ$ such that $f(x) \succ g(f(x))$.

(8) There is a reduction ordering $\succ$ such that $f(f(x)) \succ f(g(f(x)))$.

**Proposed solution.** (1) True. $g(x) \succ x$ by the subterm property, so $f(g(x)) \succ f(x)$ by compatibility with contexts.

(2) False. If $t \succ t'$ in a simplification ordering, then $\text{var}(t') \subseteq \text{var}(t)$.

(3) True. In an lpo, $f(x) \succ g(x)$ implies $f \succ g$, hence $f(x) \succ g(g(x))$.

(4) False. Choose weight 3 for $f$ and weight 2 for $g$.

(5) False. If $t \succ t'$ in a simplification ordering, then $\text{var}(t') \subseteq \text{var}(t)$.

(6) True. The variable condition for the kbo is satisfied. If $f$ has a positive weight, then the weight of the first term is larger than the weight of the second term; if $f$ has weight 0, then both terms have the same weight, and since $f(x) \succ x$, the argument tuple of the first term is lexicographically larger than the argument tuple of the second term.

(7) False. Otherwise we have $f(x) \succ g(f(x)) \succ g(g(f(x))) \succ g(g(g(f(x)))) \succ \cdots$, contradicting well-foundedness.

(8) True. The rewrite system $R = \{f(f(x)) \to f(g(f(x)))\}$ is terminating, so $\to_R^+$ is a reduction ordering with the desired property.

**Exercise 10.4** (∗)**:** Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let $h$ be a unary function symbol in $\Omega$, and let $\succ$ be precedence on $\Omega$ such that $h$ is the smallest element of $\Omega$ w.r.t. $\succ$.

Prove: For all terms $s, t \in \mathrm{T}_\Sigma(X)$, we have $s \succ_{\mathrm{lpo}} t$ if and only if $s \succeq_{\mathrm{lpo}} h(t)$.

**Proposed solution.** Since an lpo is a simplification ordering, we know that $h(t) \succ_{\mathrm{lpo}} t$, so $s \succeq_{\mathrm{lpo}} h(t)$ implies $s \succeq_{\mathrm{lpo}} h(t) \succ_{\mathrm{lpo}} t$ and thus $s \succ_{\mathrm{lpo}} t$ by transitivity. This proves the "if" part.

The "only if" part is proved by induction over $|s| + |t|$. First assume that the top symbol $f$ of $s$ is different from $h$. Since $h$ is the smallest element of the precedence, we have $f \succ h$, so $s \succ_{\mathrm{lpo}} t$ implies $s \succ_{\mathrm{lpo}} h(t)$ by Case (2b).

Otherwise $s = h(s')$.

If $h(s') \succ_{\mathrm{lpo}} t$ by Case (1), then $t$ is a variable that occurs in $h(s')$. So either $s' = t$, then $s = h(s') = h(t)$, or $s' \rhd t$, then $s' \succ_{\mathrm{lpo}} t$, and by compatibility with contexts $s = h(s') \succ_{\mathrm{lpo}} h(t)$.

If $h(s') \succ_{\mathrm{lpo}} t$ by Case (2a), then $s' = t$ or $s' \succ_{\mathrm{lpo}} t$. In the first case $s = h(s') = h(t)$, in the second case $s = h(s') \succ_{\mathrm{lpo}} h(t)$ by compatibility with contexts.

We cannot have $h(s') \succ_{\mathrm{lpo}} t$ by Case (2b) since $h$ is the smallest element of the precedence. So it remains to consider the case that $h(s') \succ_{\mathrm{lpo}} t$ by Case (2c). Then $t = h(t')$ and $s' \succ_{\mathrm{lpo}} t'$. By induction, we get $s' \succeq_{\mathrm{lpo}} h(t')$. So $s' = h(t')$ or $s' \succ_{\mathrm{lpo}} h(t')$, therefore $s = h(s') = h(h(t')) = h(t)$ or $s = h(s') \succ_{\mathrm{lpo}} h(h(t')) = h(t)$ by Case (2c).

**Exercise 10.5:** Let $\Sigma = (\{f/2,\ g/2,\ h/2\}, \emptyset)$. Let $R$ be the term rewrite system

$$\{\ g(x, f(x, y)) \to h(y, g(x, y)), \quad h(x, y) \to g(y, y)\ \}$$

Is there a lexicographic path ordering $\succ_{\mathrm{lpo}}$ such that $\to_R\ \subseteq\ \succ_{\mathrm{lpo}}$? If yes, give the precedence of this lpo; if no, explain why such an lpo does not exist.

**Proposed solution.** $\to_R$ is contained in an lpo with the precedence $f \succ h \succ g$.

**Exercise 10.6:** Let $\Sigma = (\{f/2,\ g/1,\ h/1, b/0\}, \emptyset)$. Let $R$ be the term rewrite system

$$\{\ f(g(x), y) \to g(f(x, x)),\quad h(f(x, b)) \to g(x)\ \}$$

Is there a Knuth–Bendix ordering $\succ_{\text{kbo}}$ such that $\to_R \subseteq \succ_{\text{kbo}}$? If yes, give the weights and precedence of this kbo; if no, explain why such a kbo does not exist.

**Proposed solution.** $\to_R$ is not contained in any kbo, since the first rewrite rule has more occurrences of $x$ in the right-hand side than in the left-hand side.

**Exercise 10.7:** Let $\Sigma = (\{f/1,\ g/1,\ b/0,\ c/0\}, \emptyset)$. Let $R$ be the term rewrite system

$$\{\ f(g(x)) \to g(g(f(x))),\quad c \to f(b)\ \}$$

Is there a polynomial ordering $\succ_{\mathcal{A}}$ in which the function symbols are interpreted by linear polynomials over $U_{\mathcal{A}} = \{n \in \mathbb{N} \mid n \geq 1\}$ such that $\to_R \subseteq \succ_{\mathcal{A}}$? If yes, give the polynomials by which the symbols of $\Sigma$ are interpreted; if no, explain why such an ordering does not exist.

**Proposed solution.** $\to_R$ is contained in a polynomial ordering where the symbols in $\Sigma$ are interpreted by $P_f(X_1) = 3X_1$, $P_g(X_2) = X_2 + 1$, $P_b = 1$, $P_c = 4$.

**Exercise 10.8:** Let $\Sigma = (\Omega, \emptyset)$ be a finite signature. For $t \in \mathrm{T}_\Sigma(X)$ we define $\mathrm{depth}(t) = \max\{|p| \mid p \in \mathrm{pos}(t)\}$. Let $\succ$ be a strict partial ordering on $\Omega$. The binary relation $\succ_{\text{do}}$ on $\mathrm{T}_\Sigma(X)$ is defined by: $s \succ_{\text{do}} t$ if and only if

(1) $\#(x, s) \geq \#(x, t)$ for all variables $x$ and $\mathrm{depth}(s) > \mathrm{depth}(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables $x$, $\mathrm{depth}(s) = \mathrm{depth}(t)$, and

   (a) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or

   (b) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)\ (\succ_{\text{do}})_{\text{lex}}\ (t_1, \ldots, t_m)$.

Give an example that shows that $>_{\text{do}}$ is *not* a reduction ordering.

**Proposed solution.** The relation $\succ_{\text{do}}$ is not stable under substitutions. For instance, we have $s = f(x, g(g(y))) \succ_{\text{do}} f(g(x), g(y)) = t$ since $\mathrm{depth}(s) = 3$ and $\mathrm{depth}(t) = 2$, but if $\sigma = \{x \mapsto h(h(z))\}$, then we get $s\sigma = f(h(h(z)), g(g(y))) \prec_{\text{do}} f(g(h(h(z))), g(y)) = t\sigma$ since $\mathrm{depth}(s\sigma) = 3$ and $\mathrm{depth}(t\sigma) = 4$.

**Exercise 10.9** $(*)$**:** Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let $\succ$ be a simplification ordering. Let $R$ be a TRS over $\mathrm{T}_\Sigma(X)$ such that $l \succ r$ for all $l \to r \in R$. Let $h$ be an $n$-ary function symbol in $\Omega$ (with $n > 0$) that does not occur in any left-hand side of a rule in $R$. Prove: If $R$ is confluent, then $R \cup \{h(x, \dots, x) \to x\}$ is confluent.

**Proposed solution.** First, we observe that $h(x, \dots, x)$ is larger than its proper subterm $x$ in every simplification ordering $\succ$. Therefore $l \succ r$ holds in fact for all $l \to r \in R \cup \{h(x, \dots, x) \to x\}$. Consequently, $R \cup \{h(x, \dots, x) \to x\}$ is terminating.

Second, we observe that the rewrite rule $h(x, \dots, x) \to x$ has neither a critical pair with itself, nor with any rule $l \to r \in R$ (since $h$ does not occur in $l$). Consequently, every critical pair between rules in $R \cup \{h(x, \dots, x) \to x\}$ is a critical pair between rules in $R$. Since $R$ is confluent, all critical pairs between rules in $R$ are joinable in $R$, and hence also joinable in $R \cup \{h(x, \dots, x) \to x\}$.

Using the critical pair theorem, we conclude that $R \cup \{h(x, \dots, x) \to x\}$ is locally confluent; and since it is terminating, it is also confluent.

**Exercise 10.10:** Let $\Sigma = (\{f/1,\ g/2,\ h/2,\ b/0,\ c/0\}, \emptyset)$. Let $E$ be the following set of equations over $\Sigma$:

$$f(f(x)) \approx g(b, x) \qquad (1)$$
$$h(f(y), y') \approx f(h(y, y')) \qquad (2)$$
$$g(h(z, z), c) \approx h(z, b) \qquad (3)$$

(a) Suppose that the three equations in $E$ are turned into rewrite rules by orienting them from left to right. Give all critical pairs between the resulting three rules.

(b) It is possible to orient the equations in $E$ using an appropriate kbo so that there are no critical pairs between the resulting rules. Give the weights and precedence of the kbo, and explain how the equations are oriented.

**Proposed solution.** (a) There are three critical pairs:

Between (1) at position 1 and a renamed copy of (1):
mgu $\{x \mapsto f(x')\}$,
$g(b, f(x')) \leftarrow f(f(f(x'))) \to f(g(b, x'))$,
critical pair: $\langle g(b, f(x')), f(g(b, x')) \rangle$.

Between (2) at position 1 and (1):
mgu $\{y \mapsto f(x)\}$,
$f(h(f(x), y')) \leftarrow h(f(f(x)), y') \to h(g(b, x), y')$,
critical pair: $\langle f(h(f(x), y')), h(g(b, x), y') \rangle$.

Between (3) at position 1 and (2):
mgu $\{y' \mapsto f(y),\ z \mapsto f(y)\}$,
$h(f(y), b) \leftarrow g(h(f(y), f(y)), c) \rightarrow g(f(h(y, f(y))), c)$,
critical pair: $\langle h(f(y), b), g(f(h(y, f(y))), c)\rangle$.

(b) To avoid a critical pair between (1) and itself, we must orient (1) right to left. To avoid a critical pair between (3) and (2), we must orient (2) right to left and (3) left to right.

There are several possibilities to orient the equations in this way using a kbo, for instance by choosing weight 1 for all function symbols and variables and a precedence $g \succ f \succ h \succ b \succ c$.

**Exercise 10.11:** Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$. Let $N$ be the following set of clauses over $\Sigma$:

$$P(f(x), x) \lor P(c, x) \lor R(g(x), x) \qquad (1)$$
$$\neg P(y, f(y)) \qquad (2)$$
$$\neg P(y, c) \lor \neg P(z, h(y)) \lor Q(z) \qquad (3)$$
$$Q(b) \lor Q(x) \lor \neg R(g(x), x) \qquad (4)$$
$$R(g(c), y) \qquad (5)$$

(a) Suppose that the atom ordering $\succ$ is a lexicographic path ordering with the precedence $P \succ Q \succ R \succ f \succ g \succ h \succ b \succ c$ and that the selection function $sel$ selects no literals. Compute all $\mathrm{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the conclusions of the inferences computed in part (a) is redundant w.r.t. $N$. Which one? Why?

**Proposed solution.** (a) In (1), $P(c, x)$ and $R(g(x), x)$ are not maximal since $P(f(x), x) \succ P(c, x)$ and $P(f(x), x) \succ R(g(x), x)$. In (3), $Q(z)$ is not maximal since $\neg P(z, h(y)) \succ Q(z)$. In (4), $\neg R(g(x), x)$ is not maximal since $Q(x) \succ \neg R(g(x), x)$. The remaining literals are maximal in their clauses: (1)1, (2)1, (3)1, (3)2, (4)1, (4)2, (5)1. This yields the following three inferences:

Res. (1)1, (3)1: mgu: $\{x \mapsto c,\ y \mapsto f(c)\}$
$P(c, c) \lor R(g(c), c) \lor \neg P(z, h(f(c))) \lor Q(z)$

Res. (1)1, (3)2: mgu: $\{x \mapsto h(y),\ z \mapsto f(h(y))\}$
$P(c, h(y)) \lor R(g(h(y)), h(y)) \lor \neg P(y, c) \lor Q(f(h(y)))$

Fact. (4)1, (4)2: mgu: $\{x \mapsto b\}$
$Q(b) \lor \neg R(g(b), b)$

(b) The conclusion of the first inference above contains the subclause $R(g(c), c)$, which is an instance of clause (5). Therefore, every ground instance of the conclusion follows from a smaller ground instance of (5). Hence the conclusion is redundant.

**Exercise 10.12:** Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$. Let $N$ be the following set of clauses over $\Sigma$:

$$P(x, f(x)) \lor P(x, x) \qquad (1)$$
$$\neg P(h(z), x) \lor \neg P(y, f(f(x))) \lor \neg Q(x) \lor Q(f(x)) \qquad (2)$$
$$\neg Q(h(f(x))) \lor R(h(b), y) \qquad (3)$$
$$\neg R(y, g(c)) \lor Q(g(x)) \qquad (4)$$
$$\neg Q(h(y)) \qquad (5)$$

(a) Suppose that the atom ordering $\succ$ is a Knuth–Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ Q \succ R \succ f \succ g \succ h \succ b \succ c$ and that the selection function *sel* selects no literals. Compute all $\mathrm{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the clauses (1)–(5) is redundant with respect to the others. Which one? Why? Give a brief explanation.

**Proposed solution.** (a) The following literals are maximal in clauses (1)–(5):
(1): literal 1 (literal 2 is smaller than literal 1);
(2): literals 1 and 2 (literals 3 and 4 are smaller than literal 2);
(3): literals 1 and 2;
(4): literals 1 and 2;
(5): literal 1.
From these, we get the following $\mathrm{Res}^{\succ}_{sel}$ inferences:

Resolution (1) literal 1 and (2) literal 1 (after renaming $x$ in clause (2) to $x'$):
mgu $\{x \mapsto h(z),\ x' \mapsto f(h(z))\}$,
conclusion $P(h(z), h(z)) \lor \neg P(y, f(f(f(h(z))))) \lor \neg Q(f(h(z))) \lor Q(f(f(h(z))))$.

Resolution (1) literal 1 and (2) literal 2 (after renaming $x$ in clause (2) to $x'$):
mgu $\{x \mapsto f(x'),\ y \mapsto f(x')\}$,
conclusion $P(f(x'), f(x')) \lor \neg P(h(z), x') \lor \neg Q(x') \lor Q(f(x'))$.

Resolution (3) literal 2 and (4) literal 1 (after renaming $x$ and $y$ in clause (4) to $x'$ and $y'$):
mgu $\{y' \mapsto h(b),\ y \mapsto g(c)\}$,
conclusion $\neg Q(h(f(x))) \lor Q(g(x'))$.

(b) Clause (3) is subsumed by clause (5): After applying $\sigma = \{y \mapsto f(x)\}$ to (5), the literals of (5) are a proper submultiset of the literals of (3). By Prop. 3.13.1, this means that clause (3) is redundant (i.e., every ground instance $\neg Q(h(f(t))) \vee R(h(b), t')$ of (3) is implied by a smaller ground instance $\neg Q(h(f(t)))$ of (5)).

**Exercise 10.13:** Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{f/1,\ b/0,\ c/0\}$ and $\Pi = \{P/1\}$. Suppose that the atom ordering $\succ$ is a Knuth–Bendix ordering with weight 1 for all predicate symbols, function symbols, and variables, and with the precedence $P \succ f \succ b \succ c$. Let $N = \{C_1, C_2, C_3\}$ with

$$C_1 = P(b)$$
$$C_2 = \neg P(f(f(c))$$
$$C_3 = P(x) \vee P(f(x))$$

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in $N$ looks like. How is it ordered with respect to the clause ordering $\succ_C$?

(b) Construct the candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ of the set of all ground instances of clauses in $N$. Which clauses in $G_\Sigma(N)$ are productive and what do they produce?

**Proposed solution.** (a) $G_\Sigma(N) = \{P(b)\} \cup \{\neg P(f(f(c)))\} \cup \{P(f^n(b)) \vee P(f^{n+1}(b)) \mid n \geq 0\} \cup \{P(f^n(c)) \vee P(f^{n+1}(c)) \mid n \geq 0\}$.

The clause ordering $\succ_C$ orders $G_\Sigma(N)$ in the following way:

$$\begin{array}{cc} & P(b) \\ \prec_C & P(c) \vee P(f(c)) \\ \prec_C & P(b) \vee P(f(b)) \\ \prec_C & P(f(c)) \vee P(f^2(c)) \\ \prec_C & \neg P(f^2(c)) \\ \prec_C & P(f(b)) \vee P(f^2(b)) \\ \prec_C & P(f^2(c)) \vee P(f^3(c)) \\ \prec_C & P(f^2(b)) \vee P(f^3(b)) \\ \prec_C & P(f^3(c)) \vee P(f^4(c)) \\ \prec_C & P(f^3(b)) \vee P(f^4(b)) \\ \prec_C & P(f^4(c)) \vee P(f^5(c)) \\ \prec_C & P(f^4(b)) \vee P(f^5(b)) \\ \prec_C & \vdots \end{array}$$

(b) The following table summarizes the candidate interpretation construction:

| Iter. | Clause $C$ | $R_C$ | $E_C$ |
|---|---|---|---|
| 0 | $P(b)$ | $\emptyset$ | $\{P(b)\}$ |
| 1 | $P(c) \vee P(f(c))$ | $\{P(b)\}$ | $\{P(f(c))\}$ |
| 2 | $P(b) \vee P(f(b))$ | $\{P(b), P(f(c))\}$ | $\emptyset$ |
| 3 | $P(f(c)) \vee P(f^2(c))$ | $\{P(b), P(f(c))\}$ | $\emptyset$ |
| 4 | $\neg P(f^2(c))$ | $\{P(b), P(f(c))\}$ | $\emptyset$ |
| 5 | $P(f(b)) \vee P(f^2(b))$ | $\{P(b), P(f(c))\}$ | $\{P(f^2(b))\}$ |
| 6 | $P(f^2(c)) \vee P(f^3(c))$ | $\{P(b), P(f(c)), P(f^2(b))\}$ | $\{P(f^3(c))\}$ |
| 7 | $P(f^2(b)) \vee P(f^3(b))$ | $\{P(b), P(f(c)), P(f^2(b)), P(f^3(c))\}$ | $\emptyset$ |
| 8 | $P(f^3(c)) \vee P(f^4(c))$ | $\{P(b), P(f(c)), P(f^2(b)), P(f^3(c))\}$ | $\emptyset$ |
| 9 | $P(f^3(b)) \vee P(f^4(b))$ | $\{P(b), P(f(c)), P(f^2(b)), P(f^3(c))\}$ | $\{P(f^4(b))\}$ |
| 10 | $P(f^4(c)) \vee P(f^5(c))$ | $\{P(b), P(f(c)), P(f^2(b)), P(f^3(c)), P(f^4(b))\}$ | $\{P(f^5(c))\}$ |
| 11 | $P(f^4(b)) \vee P(f^5(b))$ | $\{P(b), P(f(c)), P(f^2(b)), P(f^3(c)), P(f^4(b)), P(f^5(c))\}$ | $\emptyset$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

The candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ is $\{P(f^n(b)) \mid n \text{ is even}\} \cup \{P(f^n(c)) \mid n \text{ is odd}\}$.

**Exercise 10.14** (*)**:** Let $\Sigma = (\{f/1,\ b/0,\ c/0\},\ \{P/1\})$. Let $N$ be the following set of $\Sigma$-clauses:

$$P(b) \qquad (1)$$

$$P(f(c)) \qquad (2)$$

$$\neg P(x) \vee P(f(x)) \qquad (3)$$

Let $\succ$ be a Knuth–Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ f \succ b \succ c$. The ordering is extended to ground literals and ground clauses as usual. Give the smallest nonempty ground $\Sigma$-clauses $C_1, C_2, C_3, C_4$ such that

(a) $C_1 \in G_\Sigma(N)$ and $C_1 \in \text{Red}(N)$,

(b) $C_2 \in G_\Sigma(N)$ and $C_2 \notin \text{Red}(N)$,

(c) $C_3 \notin G_\Sigma(N)$ and $C_3 \in \text{Red}(N)$,

(d) $C_4 \notin G_\Sigma(N)$ and $C_4 \notin \text{Red}(N)$.

**Proposed solution.** $C_1 = \neg P(c) \vee P(f(c))$ is a ground instance of (3) and it is entailed by (2), which is smaller than $C_1$. In fact, $C_1$ is the only ground instance of a clause in $N$ that is redundant w.r.t. $N$.

$C_2 = P(b)$ is the smallest ground instance of a clause in $N$, namely (1). It is not entailed by smaller ground instances and therefore not redundant.

$C_3 = \neg P(c) \vee P(c)$ is a tautology and the smallest redundant clause w.r.t. $N$.

$C_4 = P(c)$ is the smallest nonempty $\Sigma$-clause. It is neither a ground instance of a clause in $N$ nor entailed by smaller ground instances (and therefore not redundant).