

Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 5: Resolution

Exercise 5.1: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1\}$. Determine for each of the following statements whether they are true or false:

- (1) There is a Σ -model $\mathcal{A}$ of $P(b) \wedge \neg P(f(b))$ such that $U_{\mathcal{A}} = \{7, 8, 9\}$.
- (2) There is a Σ -model $\mathcal{A}$ of $P(b) \wedge \neg P(f(f(b)))$ such that $f_{\mathcal{A}}(a) = a$ for every $a \in U_{\mathcal{A}}$.
- (3) $P(b) \wedge \neg P(f(b))$ has a Herbrand model.
- (4) $P(b) \wedge \forall x \neg P(x)$ has a Herbrand model.
- (5) $\forall x P(f(x))$ has a Herbrand model with a two-element universe.
- (6) $\forall x P(x)$ has exactly one Herbrand model.
- (7) $\forall x P(f(x))$ entails $\forall x P(f(f(x)))$.

Exercise 5.2: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ -formula

$$\neg P(b) \wedge P(f(f(b))) \wedge \forall x (\neg P(x) \vee P(f(x))).$$

Determine for each of the following statements whether they are true or false:

- (1) There is a Σ -model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.
- (2) There is a Σ -model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = a$ for every $a \in U_{\mathcal{A}}$.
- (3) F has exactly two Σ -models.
- (4) Every Σ -model of F is a model of $\exists x P(x)$.

- (5) Every Σ -model of F is a model of $\forall x P(f(f(x)))$.
- (6) There are infinitely many Herbrand interpretations over Σ .
- (7) There is a Herbrand model of F over Σ whose universe has exactly two elements.
- (8) There is a Herbrand model of F over Σ with an infinite universe.
- (9) F has exactly two Herbrand models over Σ .

Exercise 5.3: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{f/1, b/0, c/0\}$ and $\Pi = \{P/1\}$. Are the following statements correct?

- (1) The formula $\forall x P(x)$ has infinitely many Σ -models.
- (2) Every model of $\forall x P(x)$ is a model of $\forall x P(f(x))$.
- (3) The formula $\neg P(b) \wedge \forall x P(x)$ has a Σ -model with an infinite universe.
- (4) The formula $\neg P(b) \wedge \forall x P(f(x))$ has a Σ -model with a two-element universe.
- (5) Every Σ -model of $P(b) \wedge P(c) \wedge \forall x P(f(x))$ is a model of $\forall x P(x)$.
- (6) Every Herbrand model over Σ of $P(b) \wedge P(c) \wedge \forall x P(f(x))$ has an infinite universe.
- (7) The formula $P(b) \vee P(c)$ has exactly three Herbrand models over Σ .
- (8) The formula $\forall x P(f(x))$ has exactly four Herbrand models over Σ .

Exercise 5.4: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ -formula

$$\neg P(b) \wedge P(f(f(b))) \wedge \forall x (P(x) \vee P(f(x))).$$

Determine for each of the following statements whether they are true or false:

- (1) If $\mathcal{A}$ is a Σ -model of F , then $P_{\mathcal{A}} \neq \emptyset$ and $P_{\mathcal{A}} \neq U_{\mathcal{A}}$.
- (2) There is a Σ -model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.
- (3) There is a Σ -model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = f_{\mathcal{A}}(a')$ for all $a, a' \in U_{\mathcal{A}}$.
- (4) F has exactly four Σ -models.
- (5) There are infinitely many Herbrand interpretations over Σ .
- (6) There is an Herbrand model of F over Σ with a finite universe.
- (7) There is an Herbrand model $\mathcal{A}$ of F over Σ and an assignment β such that $\mathcal{A}(\beta)(f(b)) = \mathcal{A}(\beta)(f(f(b)))$.

Exercise 5.5: Determine for each of the following statements whether it is true or false:

- (1) If $\Sigma = (\{b/0, c/0\}, \{P/1\})$, then $P(b) \vee \neg P(c)$ has exactly three Herbrand models over Σ .
- (2) If $\Sigma = (\{f/1, c/0\}, \{P/1\})$, then $P(c) \vee P(f(c))$ has an Herbrand model over Σ whose universe has exactly four elements.
- (3) If $\Sigma = (\{f/1, c/0\}, \{P/1\})$, then $\neg P(c) \wedge \forall x P(f(x))$ has a model whose universe has exactly five elements
- (4) If $\Sigma = (\{b/0, c/0, d/0\}, \{P/1\})$, then $P(b) \vee \neg P(b)$ and $P(c) \vee \neg P(d)$ are equisatisfiable.
- (5) If $\Sigma = (\{f/1, c/0\}, \{P/1\})$, N is a set of universally quantified Σ -clauses, and every clause in N has at least one positive literal, then N has an Herbrand model.
- (6) If $\Sigma = (\{f/1, c/0\}, \{P/1\})$, N is a set of universally quantified Σ -clauses, and $N \models \neg P(x) \vee P(f(x))$, then N has a model.
- (7) If $\Sigma = (\{f/1, c/0\}, \{P/1\})$, then $\forall x P(f(x)) \models \forall y P(c) \vee P(f(f(y)))$.

Exercise 5.6: Let N be the set consisting of the following ground clauses:

$$P \vee Q \quad (1)$$

$$P \vee \neg Q \quad (2)$$

$$\neg P \vee Q \quad (3)$$

$$\neg P \vee \neg Q \quad (4)$$

- (a) Show that $N \vdash_{Res} \perp$, that is, derive $\perp$ from N using the “Resolution” and “Factorization” rules.
- (b) Why is it impossible to derive the empty clause from N without using “Factorization”?

Exercise 5.7 (*): Find a finite set N of ground clauses such that no clause in N is a tautology and such that $Res^*(N)$ is infinite.

Exercise 5.8: Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0, c/0\}$ and $\Pi = \{P/1, Q/0, R/0\}$. Use the ground resolution calculus *Res* to check whether the following clause set is satisfiable:

$$\neg P(b) \vee Q \quad (1)$$

$$\neg P(b) \vee R \quad (2)$$

$$\neg P(c) \vee Q \quad (3)$$

$$\neg Q \vee \neg R \quad (4)$$

$$Q \vee R \quad (5)$$

$$P(b) \quad (6)$$

$$\neg P(c) \quad (7)$$

Exercise 5.9: Use the ground resolution calculus to show that

$$\{(P \leftrightarrow (Q \wedge R)), (P \leftrightarrow Q)\} \models Q \rightarrow R$$

Hint: You will need some preprocessing.

Exercise 5.10: Prove or refute: $\text{Res}(N)$ is satisfiable if and only if N is satisfiable.

Exercise 5.11: Prove or refute: All clauses in $\text{Res}^*(N)$ are tautologies if and only if all clauses in N are tautologies.

Exercise 5.12 (*): Prove the following statement: If N is a set of propositional formulas and C is a propositional formula such that $N \models C$, then there exists a finite subset $M \subseteq N$ such that $M \models C$.