

Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 4: First-Order Logic

Exercise 4.1: Let $\Sigma = (\{b/0, c/0, d/0, f/1\}, \{P/1\})$. Does the formula

$$P(b) \wedge P(c) \wedge \neg P(d) \wedge \neg(\exists x P(f(f(x))))$$

have a Σ -model whose universe has exactly two elements? Give an example of such a model or show that such a model does not exist.

Exercise 4.2: Let the signature $\Sigma = (\Omega, \Pi)$ be given by $\Omega = \{+/2, s/1, 0/0\}$ and $\Pi = \emptyset$, and let

$$F_1 = \forall x (x + 0 \approx x)$$

$$F_2 = \forall x \forall y (x + s(y) \approx s(x + y))$$

$$F_3 = \forall x \forall y (x + y \approx y + x)$$

$$F_4 = \neg \forall x \forall y (x + y \approx y + x).$$

- (a) Determine a Σ -algebra $\mathcal{A}$ with an universe of exactly two elements such that $\mathcal{A}$ is a model of F_1, F_2, F_3 .
- (b) Determine a Σ -algebra $\mathcal{A}$ with an universe of exactly two elements such that $\mathcal{A}$ is a model of F_1, F_2, F_4 .

Exercise 4.3: Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1, c/0\}$. Give a Σ -model $\mathcal{A}$ of

$$\neg f(c) \approx c \quad \wedge \quad \forall x (f(f(x)) \approx x)$$

with $U_{\mathcal{A}} = \{1, 2, 3\}$.

Exercise 4.4: Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{f/2, g/2\}$ and $\Pi = \{P/2, Q/1\}$. Let

$$F = \forall x (P(x, y) \vee \exists y P(x, f(y, z)))$$

and $\sigma = \{y \mapsto g(x, z), z \mapsto g(x, y)\}$. Compute $F\sigma$.

Exercise 4.5: Let F be a formula. Prove that $\exists x F$ is satisfiable if and only if $F\{x \mapsto b\}$ is satisfiable, where b is a constant that does not occur in F .

Exercise 4.6: Let $\Sigma = (\Omega, \Pi)$ be a signature. For every Σ -formula F without equality, let $neg(F)$ be the formula that one obtains from F by replacing every atom $P(t_1, \dots, t_n)$ in F by its negation $\neg P(t_1, \dots, t_n)$ for every $P/n \in \Pi$. Prove: If F is valid, then $neg(F)$ is valid.

Hint: Somewhere in the proof you need an induction over the structure of formulas. It is sufficient if you check the base cases and $\wedge$, $\neg$, and $\exists$. The other boolean connectives and quantifiers ($\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$) can be handled analogously; you may omit them.

Exercise 4.7 (*): Let Π be a set of propositional variables. Let N and N' be sets of clauses over Π . Let S be a set of literals that does not contain any complementary literals. Prove: If every clause in N contains at least one literal L with $L \in S$ and if no clause in N' contains a literal L with $\bar{L} \in S$, then $N \cup N'$ is satisfiable if and only if N' is satisfiable.

Exercise 4.8: Let $\Sigma = (\Omega, \Pi)$ be a signature where Π contains two predicate symbols Q and R with the same arity n and possibly further predicate symbols. For any Σ -formula F let $rep(F)$ be the formula that one obtains by replacing every atom $Q(s_1, \dots, s_n)$ in F by the corresponding atom $R(s_1, \dots, s_n)$.

(a) Prove: If F is valid, then $rep(F)$ is valid. It is sufficient if you consider nonequational atoms, disjunctions $G \vee G'$, and negations $\neg G$; the other cases are handled analogously.

(b) Refute: If F is satisfiable, then $rep(F)$ is satisfiable.

Exercise 4.9 (*): Let $\Sigma = (\Omega, \Pi)$ be a signature. Let $P/1$ and $Q/0$ be predicate symbols in Π . Let N be a set of (universally quantified) clauses over Σ . Let N_0 be the set of all clauses in N that contain a literal $\neg P(t)$ for some $t \in T_\Sigma(X)$. Let N_1 be the set of all clauses in N that contain a literal $P(t')$ for some $t' \in T_\Sigma(X)$. Prove: If all clauses in $N_0 \setminus N_1$ contain also the literal $\neg Q$ and if all clauses in $N_1 \setminus N_0$ contain also the literal Q , then N and $(N \setminus N_0) \setminus N_1$ are equisatisfiable.

Exercise 4.10 (*): Let $\succ$ be a well-founded strict partial ordering on a set M . A function $\phi : M^n \rightarrow M$ with $n \geq 1$ is called strictly monotonic in the j th argument if $a_j \succ a'_j$ implies $\phi(a_1, \dots, a_j, \dots, a_n) \succ \phi(a_1, \dots, a'_j, \dots, a_n)$ for all arguments $a_1, \dots, a_n, a'_j \in M$.

(a) Prove: If the ordering $\succ$ on the set M is well-founded and total, and if $\phi : M^n \rightarrow M$ with $n \geq 1$ is strictly monotonic in the j th argument, then $\phi(a_1, \dots, a_j, \dots, a_n) \succeq a_j$ for all $a_1, \dots, a_n \in M$.

(b) In part (a), it was required that $\succ$ is a *total* ordering. Give an example that shows that the property of part (a) does not hold if the ordering $\succ$ is well-founded but not total.

(c) Use part (a) to prove the following property: Let $\Sigma = (\Omega, \Pi)$ be a signature, let $\mathcal{A}$ be a Σ -algebra. Let $\succ$ be a well-founded total ordering on the universe $U_{\mathcal{A}}$ of $\mathcal{A}$, such that $f_{\mathcal{A}} : U_{\mathcal{A}}^n \rightarrow U_{\mathcal{A}}$ is strictly monotonic in every argument for every $f/n \in \Omega$ with $n \geq 1$. Let β be an arbitrary $\mathcal{A}$ -assignment, let $t \in T_{\Sigma}(X)$. Then $\mathcal{A}(\beta)(t) \succeq \beta(x)$ for every variable $x \in \text{var}(t)$.