

Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 2: Preliminaries Continued and Propositional Logic

Exercise 2.1: Determine all strict total orderings $\succ$ on the set $\{a, b, c, d, e\}$ such that the following properties hold simultaneously:

- (1) $\{a, b\} \succ_{\text{mul}} \{a, a, c\}$
- (2) $\{c, d\} \succ_{\text{mul}} \{b, b, b\}$
- (3) $\{a, e\} \succ_{\text{mul}} \{c, e, e\}$

Exercise 2.2: Let M be a set, and let $\succ$ be a strict partial ordering over M . Let $b, b_1, b_2 \in M$, and let S, S_1, S_2 be finite multisets over M .

- (a) Prove or refute: If $\{b\} \succ_{\text{mul}} S_1$ and $\{b\} \succ_{\text{mul}} S_2$, then $\{b\} \succ_{\text{mul}} S_1 \cup S_2$.
- (b) Prove or refute: If $S \succ_{\text{mul}} \{b_1\}$ and $S \succ_{\text{mul}} \{b_2\}$, then $S \succ_{\text{mul}} \{b_1, b_2\}$.

Exercise 2.3: (a) Let $M = \{a, b, c, d\}$. Suppose that the binary relation $\rightarrow$ over multisets over M is defined by the rules (1)–(3):

- (1) $S \cup \{b, c\} \rightarrow S \cup \{a, a, a\}$
- (2) $S \cup \{b, a\} \rightarrow S \cup \{b, c, c\}$
- (3) $S \cup \{c\} \rightarrow S \cup \{d\}$

Then $\rightarrow$ can be shown to be terminating using the multiset extension $\succ_{\text{mul}}$ of an appropriate well-founded ordering on M . What does $\succ$ look like?

- (b) If the binary relation $\rightarrow$ is defined by the rules (4)–(6),

- (4) $S \cup \{a, a\} \rightarrow S \cup \{b, c\}$
- (5) $S \cup \{b, b\} \rightarrow S \cup \{a, c\}$
- (6) $S \cup \{b, c\} \rightarrow S \cup \{a, d, c, c\}$

then there is no well-founded ordering on M such that $\rightarrow$ is contained in $\succ_{\text{mul}}$. Why? Give a short explanation.

(c) Nevertheless, the relation $\rightarrow$ defined by the rules (4)–(6) is terminating. Prove it. (Hint: Think about lexicographic combinations.)

Exercise 2.4 (*): Prove: If S and S' are finite multisets over a set M , and $S \succ_{\text{mul}} S'$ holds for every strict partial ordering $\succ$ over M , then $S' \subset S$ (that is, $S' \subseteq S$ and $S' \neq S$).

Exercise 2.5: Which of the following propositional formulas are valid? Which are satisfiable?

- (1) $\neg P$
- (2) $P \rightarrow \perp$
- (3) $\perp \rightarrow P$
- (4) $(P \vee Q) \rightarrow P$
- (5) $P \rightarrow (Q \rightarrow P)$
- (6) $Q \rightarrow \neg Q$
- (7) $Q \wedge \neg Q$
- (8) $\neg(\neg P \wedge \neg\neg P)$

Exercise 2.6: Let F, G, H be propositional formulas, let p be a position of H . Prove or refute: If $H[F]_p$ is valid and $H[G]_p$ is valid, then $H[F \vee G]_p$ is valid.

Exercise 2.7: Let F, G, H be propositional formulas, let p be a position of H . Prove or refute: If $H[F \wedge G]_p$ is valid, then $H[F]_p$ and $H[G]_p$ are valid.

Exercise 2.8: Let Π be a set of propositional variables with $P, Q \in \Pi$. For every propositional formula F over Π , let $\phi(F)$ be the formula that one obtains from F by

replacing every occurrence of P by $P \vee Q$. For instance, if $F = ((R \vee \neg P) \wedge (Q \vee P))$, then $\phi(F) = ((R \vee \neg(P \vee Q)) \wedge (Q \vee (P \vee Q)))$, and if $F = R$, then $\phi(F) = R$.

(a) Prove: If $\phi(F)$ is satisfiable, then F is satisfiable. (Note: It is sufficient if you consider propositional variables, negations, and conjunctions; the other cases are treated analogously.)

(b) Refute: If $\phi(F)$ is valid, then F is valid.

Exercise 2.9: Let Π be a set of propositional variables. Let Q and R be two propositional variables in Π . For any Π -formula F let $\phi(F)$ be the formula that one obtains by replacing every occurrence of Q in F by R .

Prove: If $\phi(F)$ is satisfiable, then F is satisfiable. (It is sufficient if you consider propositional variables, conjunctions, and negations; the other cases are handled analogously.)