Retake Examination in the Course Automated Theorem Proving

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For convenience, a handout is provided with the definitions of the main calculi and concepts covered in the course.

Last name (in CAPITAL LETTERS):

First name (in CAPITAL LETTERS):

Matriculation number:

Program of study:

Hereby, I confirm the correctness of the above information:

Signature

With your signature, you confirm that you are sufficiently healthy at the beginning of the examination and that you accept the examination bindingly.

Please leave the following table blank:

Question	1	2	3	4	5	6	$\sum$
Points	20	17	12	20	15	16	100
Score							

Instructions

You have **120 minutes** at your disposal. Written or electronic aids are not permitted except for normal watches. Carrying forbidden devices, even turned off, will be considered a cheating attempt.

Write your full name and matriculation number clearly legible on this cover sheet, as well as your name in the header on each sheet. Hand in all sheets. Leave them stapled together. Use only **pens** and **neither** the color **red nor green**.

Check that you have received all the sheets and the handout. The questions can be found on **pages 3–10**. You may use the back of the sheets for secondary calculations. If you use the back of a sheet to answer, clearly mark what belongs to which question, and indicate in the corresponding question where all parts of your answer can be found. Cross out everything that should not be graded.

There are 6 questions for a total of 100 points.

Answer. This version of the exam contains model answers in blocks such as this one.

Grading. And blocks such as this one specify the grading scheme.

Question 1 (20 points)**:** Consider the first-order formula $F = \forall x, y\,((P(x) \wedge P(y)) \rightarrow P(x))$ over $\Sigma = (\{b/0\}, \{P/1\})$.

(a) Transform the negation of F into clausal normal form. Proceed one step at a time, and record each step.

Answer.

$$
\begin{array}{rl}
 & \neg\forall x, y\,((P(x) \wedge P(y)) \rightarrow P(x)) \\
\Rightarrow_P & \exists x\,\neg\forall y\,((P(x) \wedge P(y)) \rightarrow P(x)) \\
\Rightarrow_P & \exists x, y\,\neg((P(x) \wedge P(y)) \rightarrow P(x)) \\
\Rightarrow_S & \exists y\,\neg((P(c) \wedge P(y)) \rightarrow P(c)) \\
\Rightarrow_S & \neg((P(c) \wedge P(d)) \rightarrow P(c)) \\
\Rightarrow_{CNF} & \neg(\neg(P(c) \wedge P(d)) \vee P(c)) \\
\Rightarrow_{CNF} & \neg((\neg P(c) \vee \neg P(d)) \vee P(c)) \\
\Rightarrow_{CNF} & \neg(\neg P(c) \vee \neg P(d)) \wedge \neg P(c) \\
\Rightarrow_{CNF} & (\neg\neg P(c) \wedge \neg\neg P(d)) \wedge \neg P(c) \\
\Rightarrow_{CNF} & (P(c) \wedge \neg\neg P(d)) \wedge \neg P(c) \\
\Rightarrow_{CNF} & (P(c) \wedge P(d)) \wedge \neg P(c)
\end{array}
$$

Thus $N = \{P(c), P(d), \neg P(c)\}$.

Grading. 14 points:

- 4 points for prenex normal form
- 4 points for Skolemization
- 6 points for clausification

(b) Recall that $F = \forall x, y\,((P(x) \wedge P(y)) \rightarrow P(x))$ and $\Sigma = (\{b/0\}, \{P/1\})$. Use the tableau calculus with ground instantiation to prove F by refutation. Use exactly the expansion rules seen in the lecture; do not preclausify the problem. Document all your steps.

Answer.

1.	$\neg\forall x, y\,((P(x) \wedge P(y)) \rightarrow P(x))$	F's negation
2.	$\neg\forall y\,((P(c) \wedge P(y)) \rightarrow P(c))$	$1(c)[\delta]$
3.	$\neg((P(c) \wedge P(d)) \rightarrow P(c))$	$2(d)[\delta]$
4.	$P(c) \wedge P(d)$	$3_1[\alpha]$
5.	$\neg P(c)$	$3_2[\alpha]$
6.	$P(c)$	$4_1[\alpha]$

Formulas 5 and 6 are complementary.

Grading. 6 points:

- 1 point for each of the steps

Question 2 (17 points)**:** Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{zero/0,\ succ/1,\ add/2,\ mul/2\}$, let x, y be variables, and let R be the following rewrite system:

$$mul(zero, y) \to zero \quad (1)$$
$$mul(succ(x), y) \to add(y, mul(x, y)) \quad (2)$$
$$add(zero, y) \to y \quad (3)$$
$$add(succ(x), y) \to succ(add(x, y)) \quad (4)$$

(a) Specify the precedence of an lpo $\succ$ such that the rewrite rules (1)–(4) are oriented from left to right. There is no need to explain your answer.

Answer. $mul \succ add \succ succ \succ zero$.

Grading. 6 points:

- 3 points subtracted if only three out of four rewrite rules are oriented correctly

(b) Specify the precedence and the weights of a kbo $\succ$ such that the rewrite rules (1), (3), and (4) are oriented from left to right. There is no need to explain your answer.

Answer. $mul \succ add \succ succ \succ zero$ with weight 1 for all symbols and variables.

Grading. 6 points:

- 3 points subtracted if only two out of three rewrite rules are oriented correctly

(c) Compute all critical pairs between rules in R and determine whether they are joinable in R.

Answer. There are no critical pairs.

Grading. 5 points

Question 3 (12 points)**:** Refute the following set of equational clauses using the superposition calculus:

$$e \not\approx zero \qquad (1)$$
$$x \approx zero \lor div(one, x) \approx inv(x) \qquad (2)$$
$$abs(div(one, e)) \not\approx abs(inv(e)) \qquad (3)$$

Use the kbo with the precedence $abs \succ div \succ inv \succ one \succ e \succ zero$, weight 100 for $zero$, weight 10 for e, and weight 1 for all other symbols and variables. Document all of your steps.

Answer. From (2) and (1), we obtain via "Negative Superposition"

$$e \not\approx x \lor div(one, x) \approx inv(x) \qquad (4)$$

From (4) and (3), we obtain via "Negative Superposition"

$$e \not\approx e \lor abs(inv(e)) \not\approx abs(inv(e)) \qquad (5)$$

From (5), we obtain via "Equality Resolution"

$$e \not\approx e \qquad (6)$$

From (6), we obtain via "Equality Resolution" the empty clause.

Grading. 12 points:

- 3 points for each of the four inferences
  - 2 points for the conclusion
  - 1 point for the parent clauses
  - 3 points subtracted if two inferences are swapped

Question 4 (20 points)**:** Let N be the following set of propositional clauses over $\Pi = \{P, Q, R\}$:

$$\begin{array}{lllllr} \neg P & \vee & \neg Q & & & (1) \\ \neg P & \vee & \neg Q & \vee & \neg R & (2) \\ P & & & \vee & R & (3) \end{array}$$

(a) Use the DPLL procedure to find a model of N. Document each step of the procedure.

Answer. We start with the literal set $M := \emptyset$ and the clause set $N := \{(1), (2), (3)\}$. We notice that $\neg Q$ is a pure literal, so we set $M := \{\neg Q\}$. Then no more pure literals are present and no unit propagation is possible. We make an arbitrary decision and make P false, so we set $M := \{\neg Q, \neg P\}$. Next, clause (3) contains the unit literal R, so we set $M := \{\neg Q, \neg P, R\}$. At this point, all clauses in N are true in M, so we stop with $M = \{\neg Q, \neg P, R\}$ as a partial model, which is a total model as well.

Grading. 10 points:

- 3 points if the given interpretation is a (partial or total) model
- 1 point for the initial state $M = \emptyset$
- 2 points for pure literal
- 2 points for decision
- 2 points for the remaining steps

(b) Use the DPLL procedure to prove that $N \models \neg Q \vee R$. You will first need to transform the entailment into a formula to refute. Document each step of the procedure.

Answer. We use the fact that $N \models \neg Q \vee R$ if and only if $N' = N \cup \{\neg(\neg Q \vee R)\}$ is unsatisfiable. To use the DPLL procedure, we transform N' into a set of clauses and obtain $N \cup \{(4), (5)\}$, with Q (4) and $\neg R$ (5). Since (4) contains the unit literal Q, we set $M := \{Q\}$. Since (5) contains the unit literal $\neg R$, we set $M := \{Q, \neg R\}$. Since (3) contains the unit literal P, we set $M := \{Q, \neg R, P\}$. At this point, (1) is false in M, and there is nowhere to backtrack to, so the clause set N' is unsatisfiable.

Grading. 10 points:

- 2 points for N'
- 2 points for clausification of N'
- 2 points for each of the three unit propagations

Question 5 (15 points)**:** For this question, we use the signature $\Sigma = (\{f/1, g/1, b/0\}, \{P/1\})$, the clause set $N = \{P(g(x))\}$ over Σ, and the kbo with $P \succ g \succ f \succ b$ and weights 1 for all symbols and variables.

(a) Complete the following table summarizing the first five iterations of the candidate interpretation construction. Hint: Recall that the clauses considered in the second column must be groundings of clauses in N.

Iter.	Clause C	I_C	Δ_C
0	$P(g(b))$	$\emptyset$	$\{P(g(b))\}$
1			
2			
3	$P(g(f(f(b))))$		
4			

Answer.

Iter.	Clause C	I_C	Δ_C
0	$P(g(b))$	$\emptyset$	$\{P(g(b))\}$
1	$P(g(f(b)))$	$\{P(g(b))\}$	$\{P(g(f(b)))\}$
2	$P(g(g(b)))$	$\{P(g(b)), P(g(f(b)))\}$	$\{P(g(g(b)))\}$
3	$P(g(f(f(b))))$	$\{P(g(b)), P(g(f(b))), P(g(g(b)))\}$	$\{P(g(f(f(b))))\}$
4	$P(g(f(g(b))))$	$\{P(g(b)), P(g(f(b))), P(g(g(b))), P(g(f(f(b))))\}$	$\{P(g(f(g(b))))\}$

Grading. 11 points:

- 1 point for each of the table entries, acknowledging Folgefehler

(b) Determine I_∞ for the clause set N.

Answer. Extrapolating from the answer to part (a), we conclude that all clauses of the form $P(g(t))$ are productive, where t is a ground term. Thus $I_\infty = \{P(g(t)) \mid t \in \mathrm{T}_\Sigma\}$.

Grading. 4 points:

- 2 points if I_∞ is a model
- 2 points if I_∞ is correctly defined

Question 6 (16 points)**:** We call a ground clause *symmetric* if for every literal L it contains, it also contains the complementary literal $\overline{L}$. Thus, $P(b) \lor Q(c) \lor \neg P(b) \lor \neg Q(c)$ is symmetric.

(a) Prove that every inference of the ground resolution calculus from symmetric premises generates a symmetric conclusion.

Answer. For "Resolution," consider the premises $D \lor A$ and $C \lor \neg A$. Since $D \lor A$ is symmetric, $\neg A$ must occur in D, and since $C \lor \neg A$ is symmetric, A must occur in C. For each literal in D, the conclusion $C \lor D$ contains its complement because D does, and for each literal in C, the conclusion $C \lor D$ contains its complement because C does. Moreover, both A and $\neg A$ must occur in $C \lor D$. Hence every literal in $C \lor D$ is accompanied by its complement, and thus $C \lor D$ is symmetric.

For "Factorization," we note that symmetry is invariant over the number of duplicate literals. For example, $P \lor \neg P$ is just as much symmetric as $P \lor P \lor \neg P$, and removing a duplicate, as "Factorization" does, has no impact on symmetry.

Grading. 6 points:

- 4 points for "Resolution"
- 2 points for "Factorization"

(b) Exhibit a nonempty set of symmetric clauses that is satisfiable. Briefly explain why the set is satisfiable.

Answer. The set $\{P \vee \neg P\}$ is satisfiable. The clause is a tautology.

Grading. 5 points:

- 3 points for the set
- 2 points for the explanation

(c) Exhibit a set of symmetric clauses that is unsatisfiable. Briefly explain why the set is unsatisfiable.

Answer. The set $\{\bot\}$ is unsatisfiable. The empty clause is a contradiction.

Grading. 5 points:

- 3 points for the set
- 2 points for the explanation