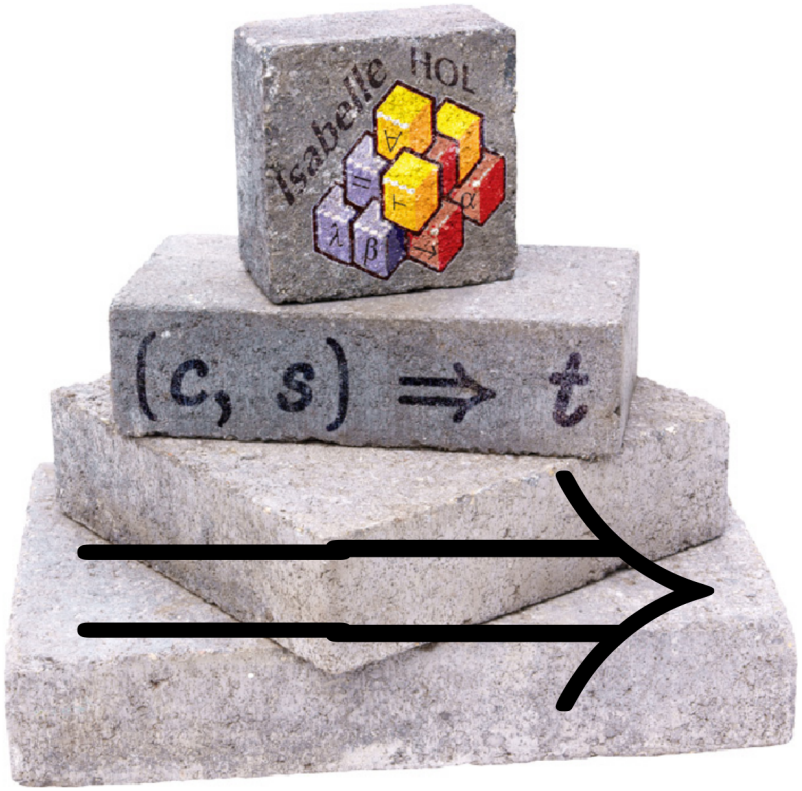

Higher-Order Logics and Interactive Theorem Proving with Isabelle/HOL II

Formal Systems II: Application

Bernhard Beckert · Mattias Ulbrich · Michael Kirsten | Summer Term 2025

More on Function Definitions

Non-recursive definitions

- **definition** $sq :: nat \Rightarrow nat$ **where** $sqn = n * n$
- No pattern matching, just $fx_1 \ldots x_n = \ldots$

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More on Function Definitions

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(Possibly) recursive functions: fun

- Pattern-matching over datatype constructors
- Order of equation matters
- Termination must be provable automatically by size measures
- Proves customized induction schema
- Example: **fun** $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$$ack\ 0\ n = Suc\ n \mid$$
$$ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0) \mid$$
$$ack\ (Suc\ m)\ (Suc\ n) = ack\ m\ (ack\ (Suc\ m)\ n)$$

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Proof method *simp*

Goal: 1. $[\![P_1; \ldots; P_m]\!] \Longrightarrow C$

apply$(simp\ add\text{:}\ eq_1\ \ldots\ eq_n)$

Simplify $P_1\ \ldots\ P_m$ and C using

- lemmas with attribute $simp$
- rules from **fun** and **datatype**
- additional lemmas $eq_1\ \ldots\ eq_n$
- assumptions $P_1\ \ldots\ P_m$

Variations:

- $(simp\ \ldots\ del\text{:}\ \ldots)$ removes $simp$-lemmas
- add and del are optional

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auto versus *simp*

- $auto$ acts on all subgoals
- $simp$ acts only on subgoal 1

- $auto$ applies $simp$ and more

- $auto$ can also be modified:
  $(auto\ simp\ add\text{: } \ldots\ \ simp\ del\text{: } \ldots)$

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Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add\text{: } f\_def \ldots)$$

f is the function whose definition is to be unfolded.

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Case Splitting with *simp/auto*

Automatic:

$$P\ (\mathit{if}\ A\ \mathit{then}\ s\ \mathit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

By hand:

$$P\ (\mathit{case}\ e\ \mathit{of}\ 0 \Rightarrow a \mid \mathit{Suc}\ n \Rightarrow b)$$
$$=$$
$$(e = 0 \longrightarrow P(a)) \wedge (\forall n.\ e = \mathit{Suc}\ n \longrightarrow P(b))$$

Proof method: $(\mathit{simp}\ \mathit{split}\colon\ \mathit{nat.split})$
Or auto. Similar for any datatype t: $\mathit{t.split}$

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More Proof Methods

intro ⟨*intro-rules*⟩	repeatedly applies intro rules
elim ⟨*elim-rules*⟩	repeatedly applies elim rules
clarify	applies all safe rules that do not split the goal
safe	applies all safe rules
blast	an automatic tableaux prover (works well on predicate logic)
fast	another automatic search tactic
fastforce	rewriting, logic, sets, relations and a bit of arithmetic
arith	proves linear formulas (no "$*$"), complete for quantifier-free real and Presburger arithmetic

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https://www21.in.tum.de/teaching/fds/SS22/assets/Demos/Simp_Demo.thy

$\Longrightarrow$ versus $\longrightarrow$

$\Longrightarrow$ is part of the Isabelle framework. It structures theorems and proof states: $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$

$\longrightarrow$ is part of HOL and can occur inside the logical formulas A_i and A.

Phrase theorems like this $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$

not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

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Sets over type *'a*

$$'a\ set$$

- $\{\}, \quad \{e_1, \ldots, e_n\}$
- $e \in A, \quad A \subseteq B$
- $A \cup B, \quad A \cap B, \quad A - B, \quad - A$
- $\ldots$

$\in$	`\<in>` :
$\subseteq$	`\<subseteq>` `<=`
$\cup$	`\<union>` `Un`
$\cap$	`\<inter>` `Int`

Set Comprehension

- $\{x.\ P\}$ where x is a variable
- But not $\{t.\ P\}$ where t is a proper term
- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

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Isar Core Syntax

proof = **proof** [method] step* **qed**
| **by** method

method = $(simp \ldots)$ | $(blast \ldots)$ | $(induction \ldots)$

step = **fix** variables $(\bigwedge)$
| **assume** prop $(\Longrightarrow)$
| [**from** fact+] (**have** | **show**) prop proof

prop = [name:] "formula"

fact = name | ...

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Example: Cantor's Theorem

lemma $\neg\ surj(f :: 'a \Rightarrow 'a\ set)$
proof default proof: assume $surj$, show $False$
  assume a: $surj\ f$
  from a **have** b: $\forall\ A.\ \exists\ a.\ A = f\ a$
    by$(simp\ add:\ surj\_def)$
  from b **have** c: $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$
    by $blast$
  from c **show** $False$
    by $blast$
qed

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https://www21.in.tum.de/teaching/fds/SS22/assets/Demos/Isar_Demo.thy

Abbreviations

$this$	$=$	the previous proposition proved or assumed
then	$=$	**from** $this$
thus	$=$	**then show**
hence	$=$	**then have**

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using and with

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

with facts

=

from facts *this*

Structured Lemma Statement

lemma
  fixes $f :: 'a \Rightarrow 'a\ set$
  assumes s: $surj\ f$
  shows $False$
proof $-$ no automatic proof step
  have $\exists\ a.\ \{x.\ x \notin f\ x\} = f\ a$ **using** s
    by$(auto\ simp$: $surj\_def)$
  thus $False$ **by** $blast$
qed

Proves $surj\ f \Longrightarrow False$
but $surj\ f$ *becomes local fact* s *in proof.*

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The Essence of Structured Proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

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Structured Lemma Statements

fixes $x :: \tau_1$ **and** $y :: \tau_2$ ...
assumes *a:* P **and** *b:* Q ...
shows R

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

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Case Distinction

show R
proof $cases$
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume $\neg$ P
  $\vdots$
  show R $\langle proof \rangle$
qed

have $P \vee Q$ $\langle proof \rangle$
then show R
proof
  assume P
  $\vdots$
  show R $\langle proof \rangle$
next
  assume Q
  $\vdots$
  show R $\langle proof \rangle$
qed

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Set Equality and Subset

show $A = B$
proof
  show $A \subseteq B$ $\langle proof \rangle$
next
  show $B \subseteq A$ $\langle proof \rangle$
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  $\vdots$
  show $x \in B$ $\langle proof \rangle$
qed

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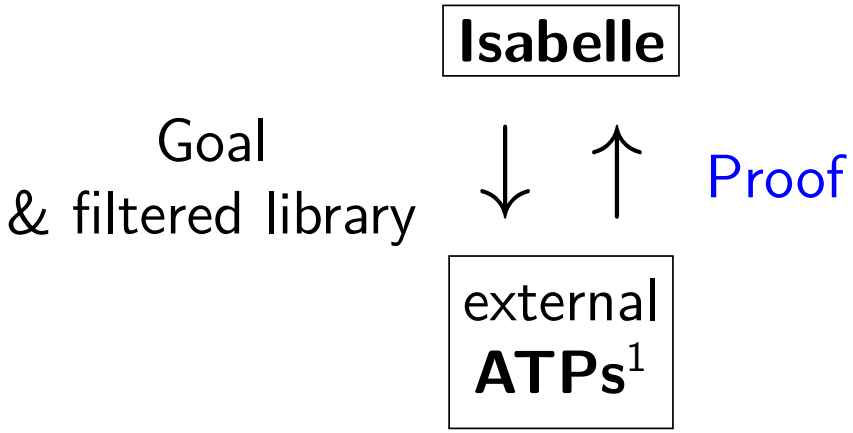
Sledgehammer

Architecture:



Characteristics:

- Sometimes it works,
- sometimes it doesn't.

Do you feel lucky?

[1] Automatic Theorem Provers

Sledgehammer

Sledgehammer:

- ➜ *Connects Isabelle with ATPs and SMT solvers:*
  E, SPASS, Vampire, CVC3, Yices, Z3
- ➜ *Simple invocation:*
  - ➜ *Users don't need to select or know facts*
  - ➜ *or ensure the problem is first-order*
  - ➜ *or know anything about the automated prover*
- ➜ *Exploits local parallelism and remote servers*

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Download

https://www21.in.tum.de/teaching/fds/SS22/assets/Demos/Auto_Proof_Demo.thy

Download `Chapter2.thy` from
`http://concrete-semantics.org/Exercises/templates.tar`
and try to do some more exercises.

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