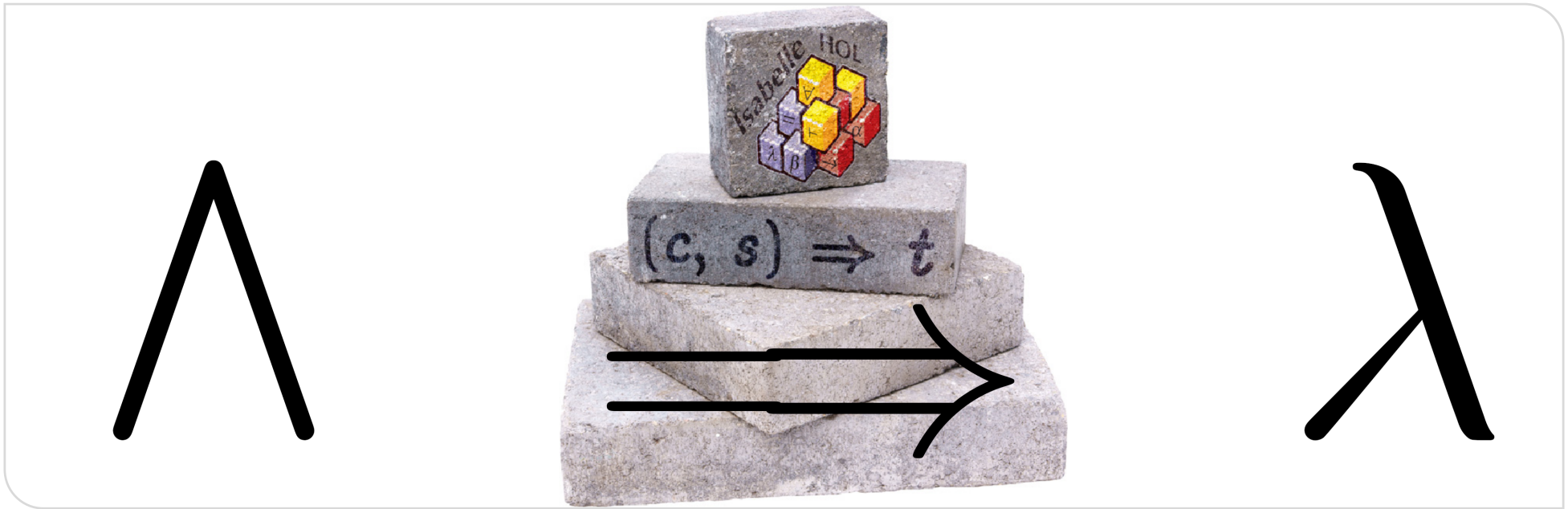
Karlsruhe Institute of Technology

KASTEL

Formal Systems II: Application

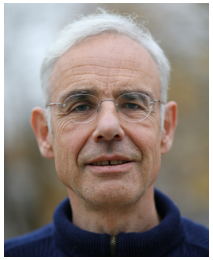
Higher-Order Logics and Interactive Theorem Proving with Isabelle

Bernhard Beckert · Mattias Ulbrich · Michael Kirsten · Alexander Weigl | SS 2023



Credits

Most material (originally) shamelessly stolen from



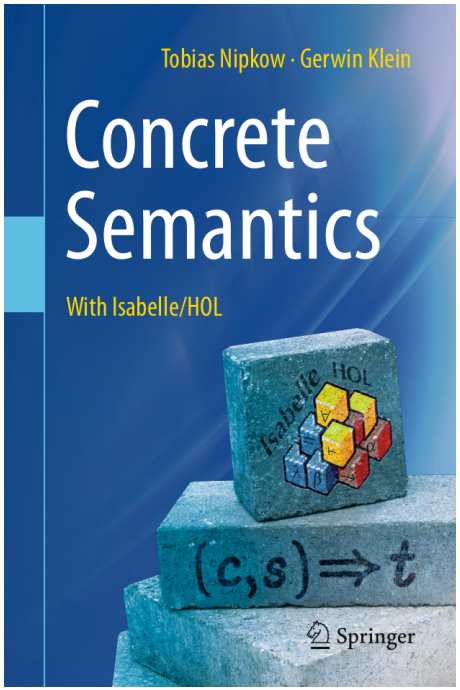
Tobias Nipkow, Lawrence Paulson, Makarius Wenzel



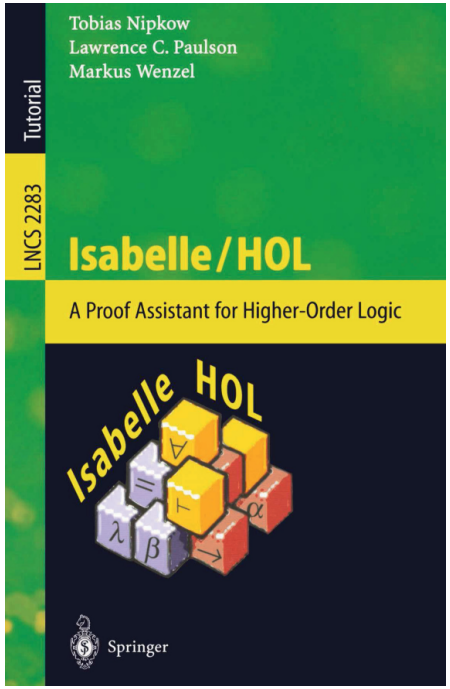
Gerwin Klein, John Harrison, Christian Sternagel

Don’t blame them, errors are mine!

Literature

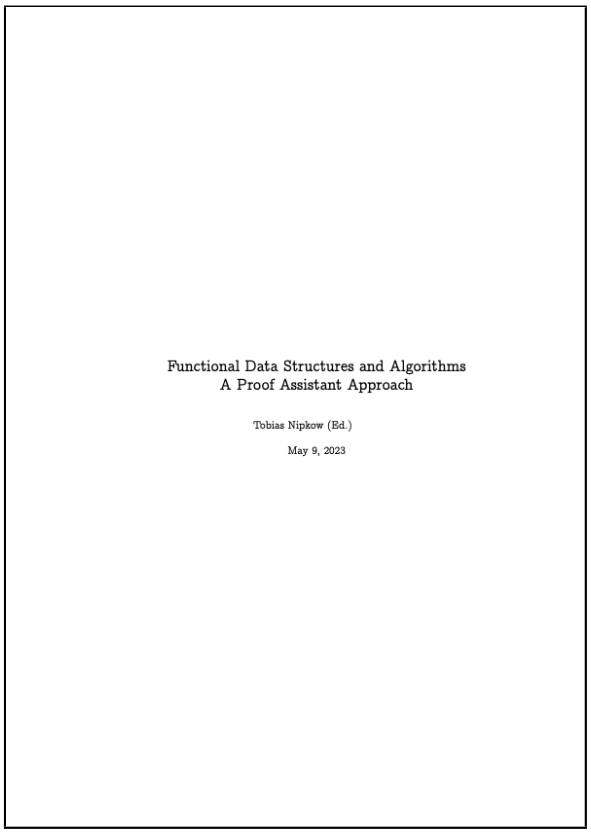


Tobias Nipkow and Gerwin Klein:
Concrete Semantics:
With Isabelle/HOL
Springer International Publishing, 2014
`http://concrete-semantics.org/`



Tobias Nipkow and Markus Wenzel:
Isabelle/HOL:
A Proof Assistant for
Higher-Order Logic
Lecture Notes in Computer Science
Springer-Verlag, 2002

Literature (cont'd)


Functional Data Structures and Algorithms
A Proof Assistant Approach

Tobias Nipkow (Ed.)

May 9, 2023

Tobias Nipkow (Ed.):
Functional Data Structures and Algorithms: A Proof Assistant Approach
Under Development, 2023
`https://functional-algorithms-verified.org`

This book is meant to evolve over time. If you would like to contribute, get in touch!

What is a Theorem Prover?

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There are other (algorithmic) verification tools:

- Model checking, static analysis, ...
- Usually do not deliver proofs
- See further topics of this course

Some Impressive Verifications

C compiler (CompCert)

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Won *2021 ACM Software System Award*

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Usually balancing simplicity against flexibility/expressiveness

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Type theory (more computer science interconnections)

- Simple type theory: HOL family and Isabelle/HOL
- Martin-Löf type theory: Agda, Nuprl
- Calculus of inductive constructions: Coq
- Other typed formalisms: IMPS, PVS

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Others

- Primitive recursive arithmetic (no explic. quantifiers): ACL2, NQTHM
- Homotopy type theory with experimental implementations

Software Architecture

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- **de Bruijn approach:**
  Generate proofs that can be certified by a simple, separate checker
- **LCF approach (orig. *Logic of Computable Functions*):**
  Reduce all rules to sequences of primitive inferences implemented by a small logical *kernel*
  - Specification methods and automatic proof procedures expand to full proofs
  - Unsoundness less likely
  - Implementation more complicated, performance can suffer
  - Examples: Isabelle, HOL, Coq

Higher-Order Logic (HOL)

- FOL extended with functions and sets, i.e.,
  functions are values, too!
- Polymorphic types, incl. truth value type
- No distinction between terms and formulas
- ML-style functional programming

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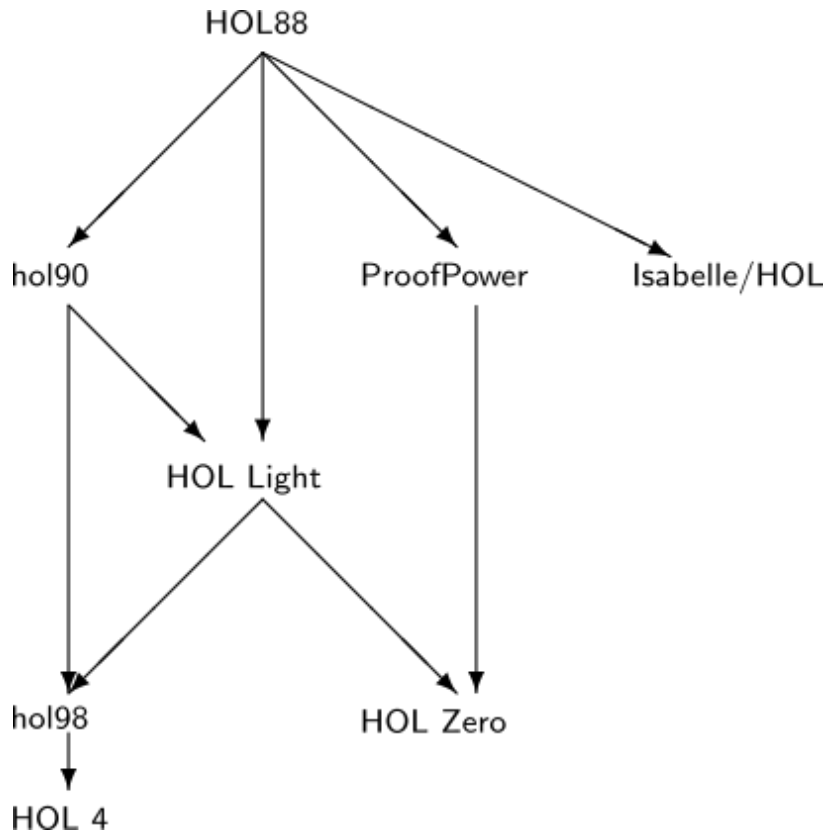

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More than just yes/no, you can interactively guide the system

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Not specialized to one particular logic
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More than just yes/no, you can interactively guide the system

Proof Assistant:
Given proof structure, helps to explore, find, and maintain proofs by checking correctness of each step

System Architecture

Isabelle – generic, interactive theorem prover

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Standard ML – logic implemented as ADT

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Prover IDE (jEdit) – user interface

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User can access all layers!

System Architecture

Prover IDE (jEdit) – user interface	Isabelle/PIDE, Isabelle/jEdit
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System Requirements

- **Linux**, **Windows**, or **MacOS X** (10.8 +)
- **Standard ML** (PolyML implementation)
- **Java** (for jEdit)
- Download from `https://isabelle.in.tum.de` with lots of documentation

Let us start with the basics . . .

Which of the following three formulas have the same meaning?

1. $A \Longrightarrow (B \Longrightarrow C)$
2. $(A \Longrightarrow B) \Longrightarrow C$
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Let us start with the basics ...

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Notation:

- $A \Longrightarrow (B \Longrightarrow C)$ means $A \Longrightarrow B \Longrightarrow C$ means $[\![A; B]\!] \Longrightarrow C$
- $\dfrac{A_1 \qquad \cdots \qquad A_n}{C}$ means $A_1 \Longrightarrow \cdots \Longrightarrow A_n \Longrightarrow C$

Basic Syntax

Types:

$$\tau ::= (\tau)$$

$\mid$	$bool \mid nat \mid int \mid \ldots$	base types
$\mid$	$'a \mid 'b \mid \ldots$	type variables
$\mid$	$\tau \Rightarrow \tau$	total functions
$\mid$	$\tau \times \tau$	pairs (ascii: *)
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$\ldots$ must be well-typed!

Some More Basics

- Automatic β-reduction, e.g., $(\lambda x.\ x + 5)\ 3 = 3 + 5$

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  (Enclose in parentheses!)

Theory = Isabelle Module

Syntax:

```
theory MyTh
imports T1 ... Tn
begin
(definitions, theorems, proofs, ...)*
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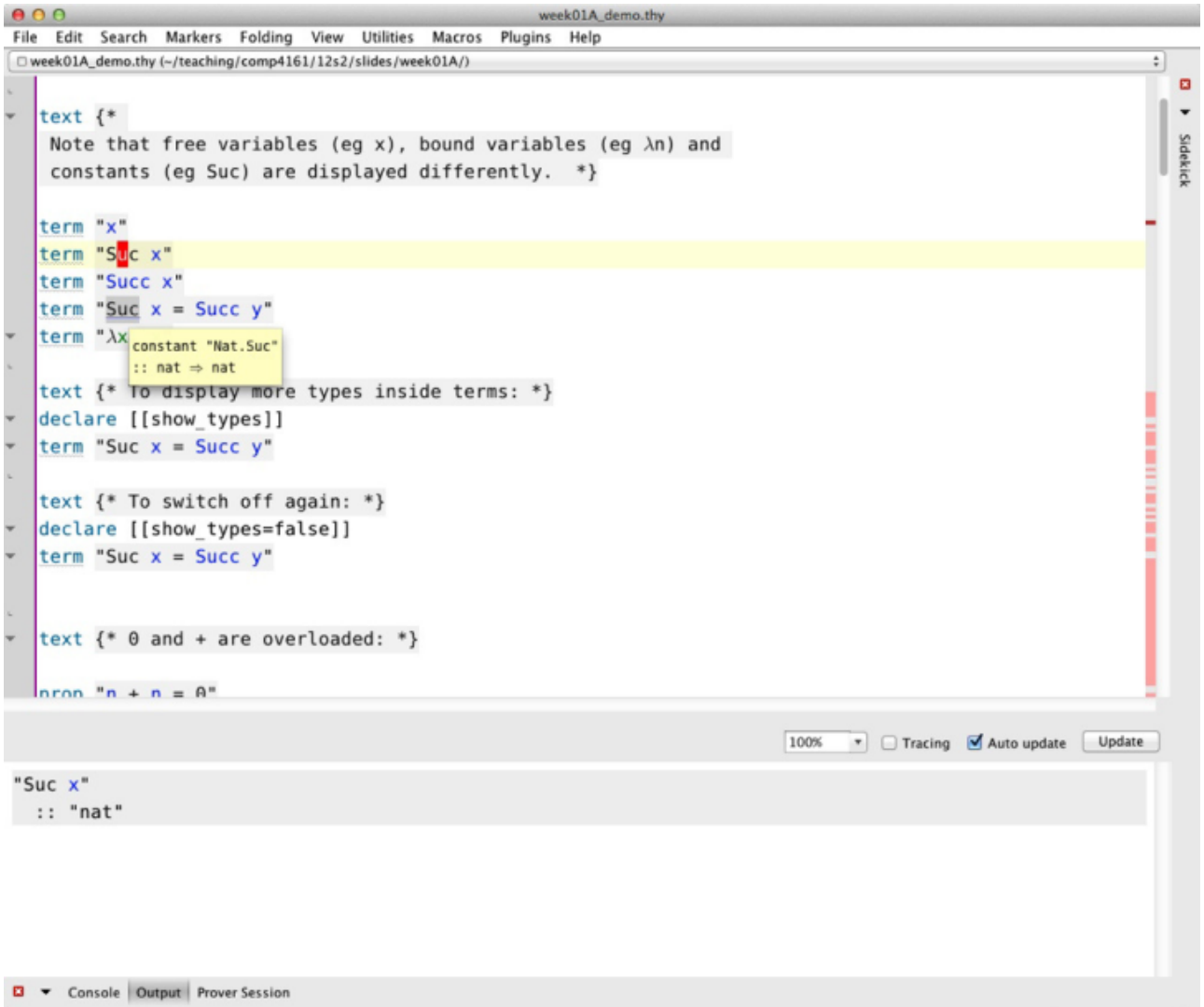
Usually: `imports Main`

Download

https://www21.in.tum.de/teaching/fds/SS22/assets/Demos/Overview_Demo.thy

Isabelle jEdit/PIDE



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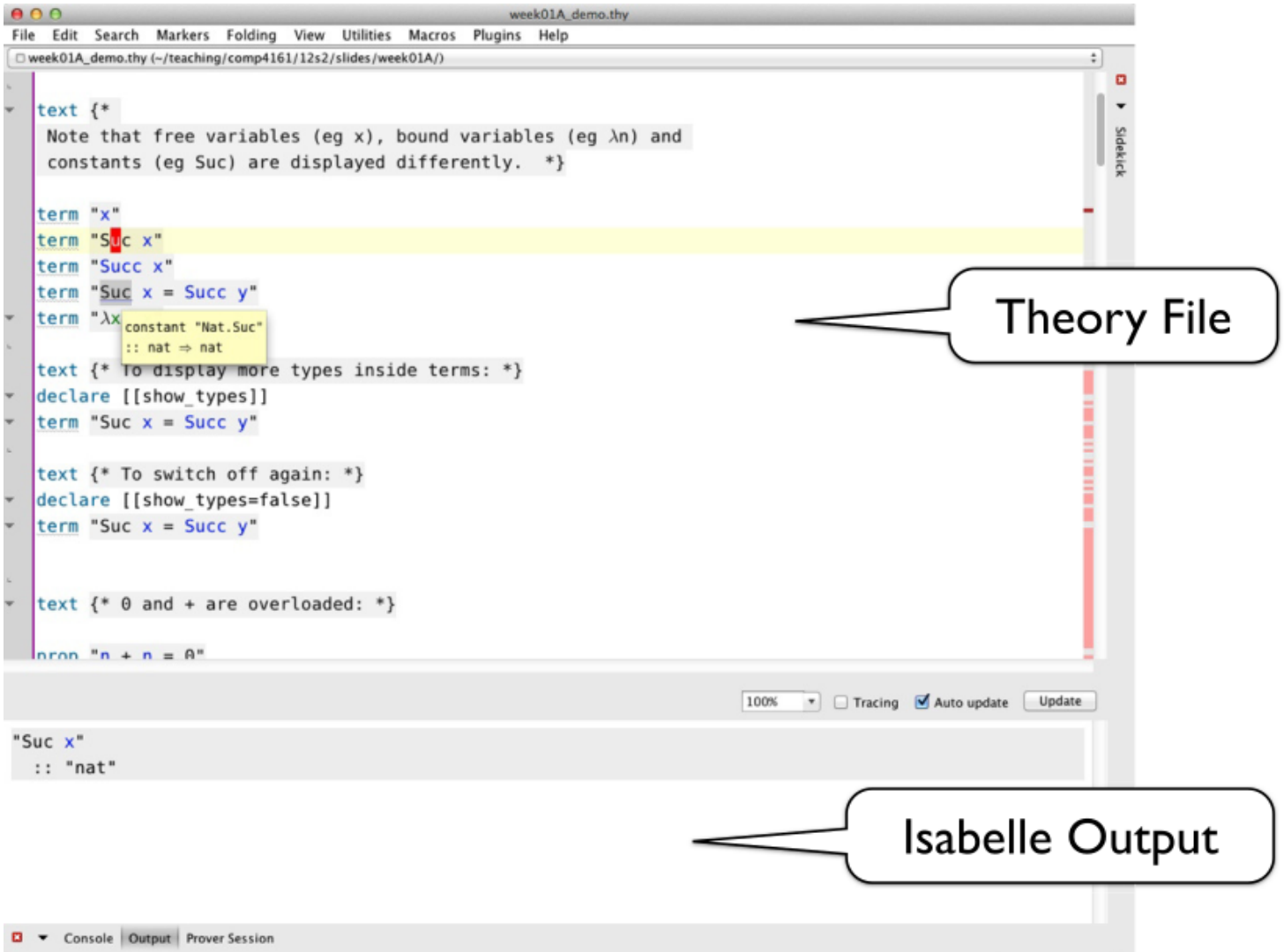

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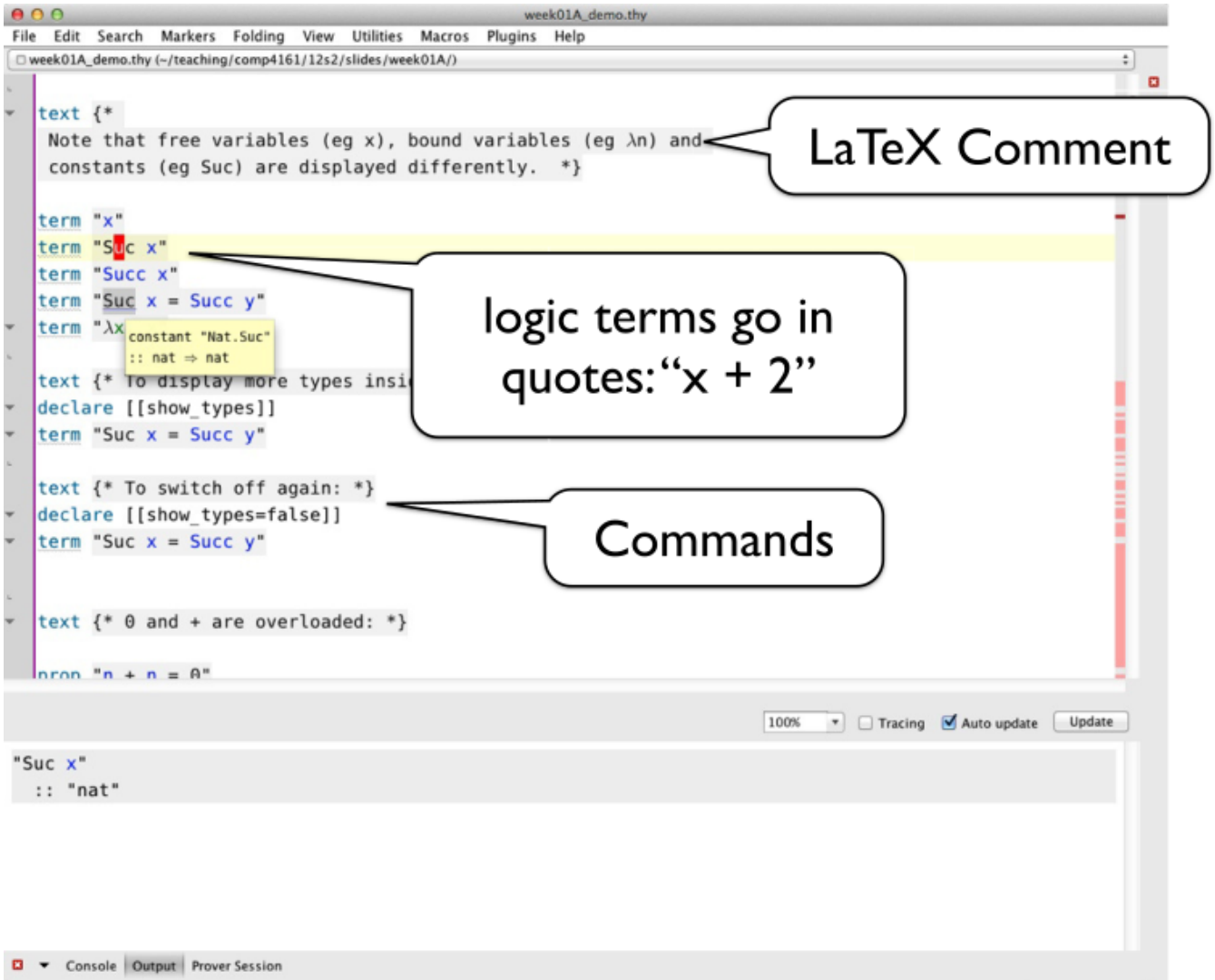

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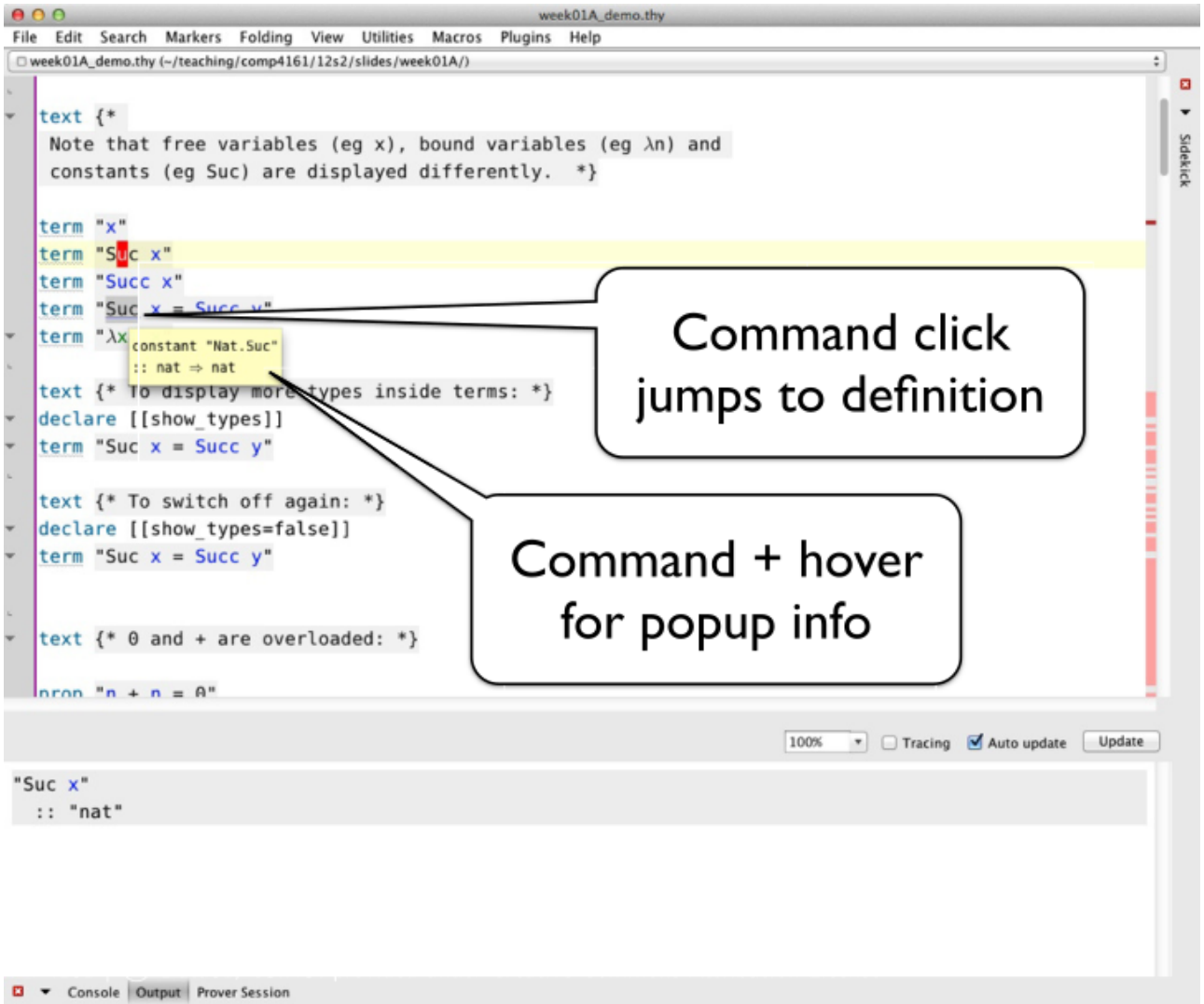

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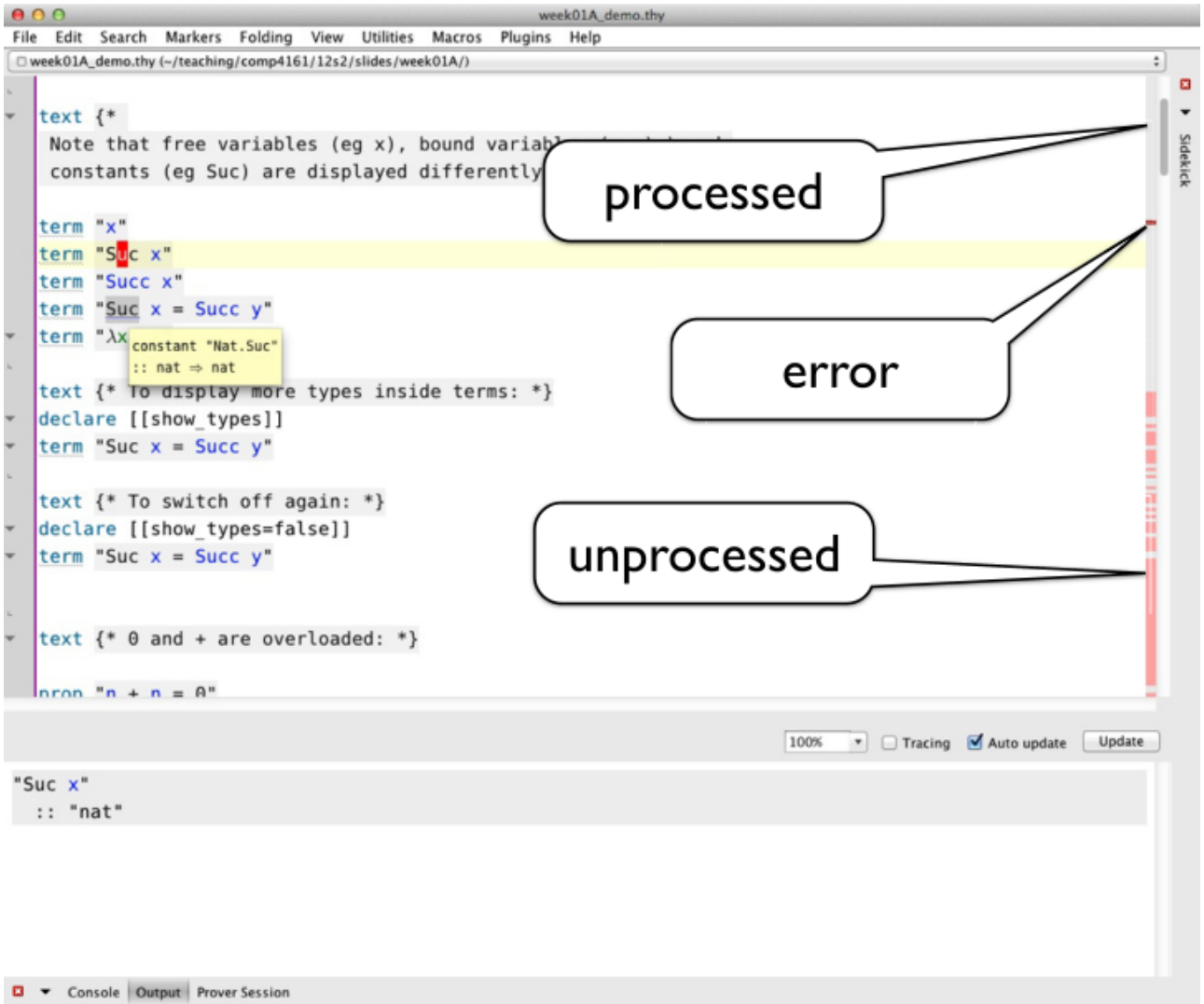

Exploring a Theory in Isabelle

Some Diagnostic Commands

find_theorems $\langle args \rangle$	print all theorems matching $\langle args \rangle$
print_cases	print currently available cases
prop $\langle formula \rangle$	print proposition $\langle formula \rangle$
term $\langle term \rangle$	print term $\langle term \rangle$ and its type
thm $\langle name \rangle$	print theorem called $\langle name \rangle$
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Three Kinds of Variables

- Free variables (blue in Isabelle/jEdit)
- Bound variables (green in Isabelle/jEdit)
- Schematic variables (dark blue with leading ? in Isabelle/jEdit); can be replaced by arbitrary values

Type *nat*

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Numbers and arithmetic operations are overloaded:
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You need type annotations: $1 :: nat$, $x + (y :: nat)$
unless the context is unambiguous: $Suc\ z$

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Types, Functions, Lemmas, Proof Methods

- datatype defines (possibly) recursive data types.
- fun defines (possibly) recursive functions by pattern-matching over datatype constructors.
- lemma or theorem defines a theorem (that has to be proven), either as a single formula or broken into fixed variables (**fixes**), assumptions (**assumes**), and proof obligation (**shows**)

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Proof Methods

- *induction* performs structural induction on some variable (if the type of the variable is a datatype).
- *auto* solves as many subgoals as it can, mainly by simplification (symbolic evaluation):
  "=" is used only from left to right!

Proofs

General Schema:

lemma *name*: "..."
apply (...)
apply (...)
⋮
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The Proof State: $\bigwedge x_1 \ldots x_p.\ A \Longrightarrow B$

$x_1 \ldots x_p$	fixed local variables
A	local assumption(s)
B	actual (sub)goal

Proofs

General Schema: for apply scripts

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If the lemma is suitable as a simplification rule: **lemma** *name*`[simp]`: "..."

The Proof State: $\bigwedge x_1 \ldots x_p.\ A \Longrightarrow B$

$x_1 \ldots x_p$	fixed local variables
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Unreadable, hard to maintain, and does not scale!

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Proofs Using Isabelle/Isar

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But: **apply** still useful for proof exploration

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Lists of elements of type $'a$

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Syntactic Sugar:

- $[] = \ Nil$ empty list
- $x\ \#\ xs = \ Cons\ x\ xs$: list with first element x ("*head*") and rest xs ("*tail*")
- $[x_1, \ldots, x_n] = \ x_1\ \#\ldots x_n\ \#\ []$

Structural Induction for Lists

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$$\frac{P([]) \qquad \bigwedge x\ xs.\ P(xs) \Longrightarrow P(x \# xs)}{P(xs)}$$

Download

https://www21.in.tum.de/teaching/fds/SS22/assets/Demos/List_Demo.thy

Download `Chapter2.thy` from

http://concrete-semantics.org/Exercises/templates.tar

and try to (re-)do exercises up to summation formula.