Adding Sorts to an Isabelle Formalization of Superposition

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LUDWIG-
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Superposition

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until a contradiction is found or saturation is reached.

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Clauses: $l_1 \vee \cdots \vee l_n$

Superposition

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E, SPASS, Vampire, and Zipperposition implement superposition.

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Assume $\pi \neq 0$.
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Then the absolute value of the inverse of π is the absolute value of $1/\pi$.

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$$\downarrow$$

$$\mathsf{pi} \not\approx \mathsf{zero} \ \wedge\ (\forall x.\ x \not\approx \mathsf{zero} \Rightarrow \mathsf{inv}(x) \approx \mathsf{div}(\mathsf{one}, x))$$

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$$\mathsf{pi} \not\approx \mathsf{zero} \qquad x \approx \mathsf{zero} \vee \mathsf{div}(\mathsf{one}, x) \approx \mathsf{inv}(x)$$

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$$\bot$$

Formalization

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We formalized soundness and completeness using the saturation framework (Waldmann et al. 2022; Tourret and Blanchette 2021).

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locale superposition_calculus =

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for

$\prec_t :: {}'t \Rightarrow {}'t \Rightarrow \mathit{bool}$ **and**

select :: $'t\ \mathit{clause} \Rightarrow {}'t\ \mathit{clause}$ **and** ...

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locale superposition_calculus =
  nonground_order ≺ₜ +                          ⎫
                                                ⎬ assumptions
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  for
    ≺ₜ :: 't ⇒ 't ⇒ bool and
    select :: 't clause ⇒ 't clause and ...
begin

inductive superposition :: 't clause ⇒ 't clause ⇒ 't clause ⇒ bool where ...

inductive eq_resolution :: 't clause ⇒ 't clause ⇒ bool where ...

inductive eq_factoring :: 't clause ⇒ 't clause ⇒ bool where ...

end
```

Adding Types

Why Do We Want Types?

$$x \approx y \qquad \neg \mathsf{p}$$

The clause set is satisfiable in any interpretation with a single-element domain.

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The constant t encodes truth.

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The clause set is satisfiable if the types of x and y are not Boolean.

Modern provers support types natively.

Simple Monomorphic Types – Example

$$\frac{}{\mathcal{F}, \mathcal{V} \vdash \mathsf{replicate}\ x\ \mathsf{five} : \mathit{String}}$$

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$$\frac{\mathcal{F}\ \textsf{replicate}\ 2 = ([\mathit{Char}, \mathit{Nat}],\ \mathit{String})}{\mathcal{F}, \mathcal{V} \vdash \textsf{replicate}\ x\ \textsf{five} : \mathit{String}}$$

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Simple Monomorphic Types – Example

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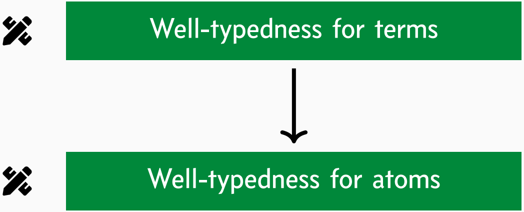
Liftings

Well-typedness for terms

Liftings



Well-typedness for terms

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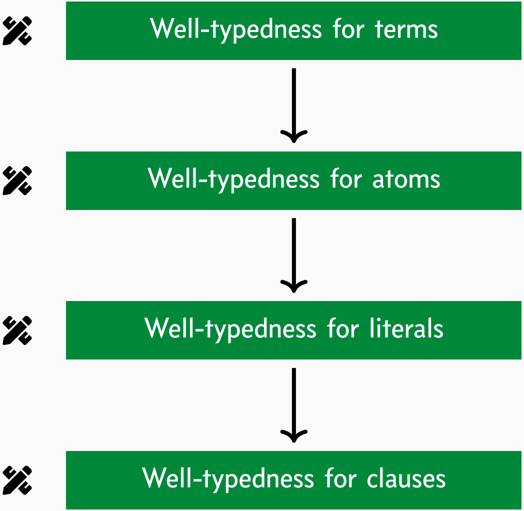

Well-typedness for terms

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Well-typedness for literals

Well-typedness for clauses

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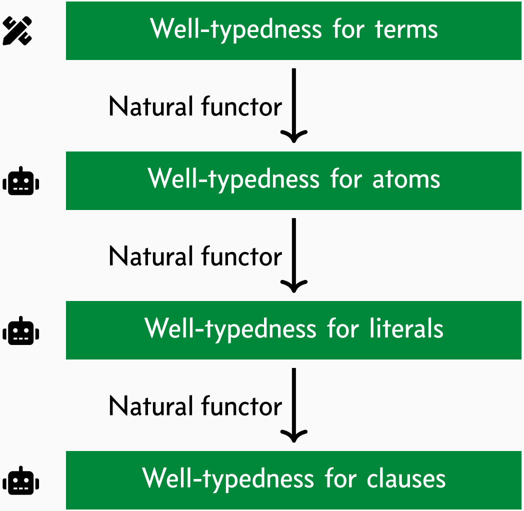

✎ Well-typedness for terms

↓ Natural functor

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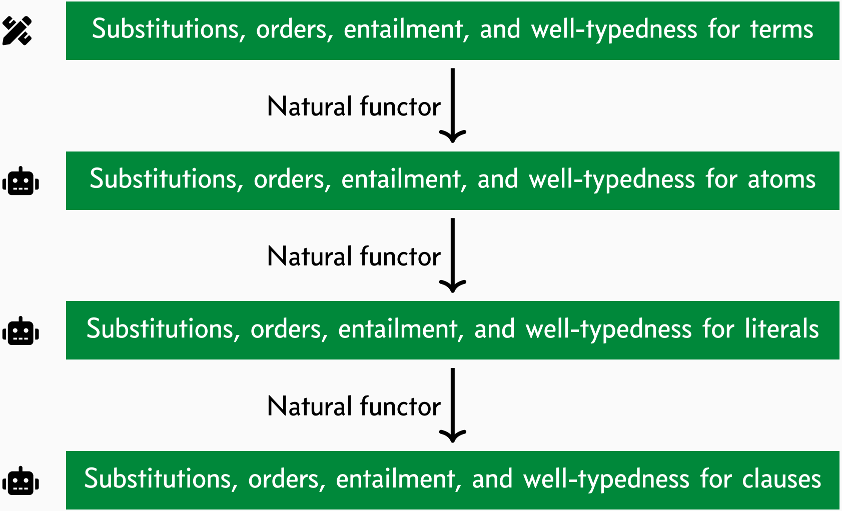

✎ Substitutions, orders, entailment, and well-typedness for terms

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locale superposition_calculus $=$
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begin
  ...

inductive eq_resolution :: ('t, 'v, 'ty) typed_clause ⇒ ('t, 'v, 'ty) typed_clause ⇒ bool where
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Completeness Proof

Theorem

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Completeness Proof

Theorem

For every set N that is saturated, if N entails $\bot$*, then* $\bot \in N$.

$$\begin{tikzcd}
N \arrow[d, "\text{ground clauses}"'] & \\
N_G \arrow[r, "\text{construct model}"'] & M_G
\end{tikzcd}$$

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For every set N that is saturated, if N entails $\bot$, then $\bot \in N$.

$$\begin{tikzcd}
N \arrow[d, "\text{ground clauses}"'] & & M \\
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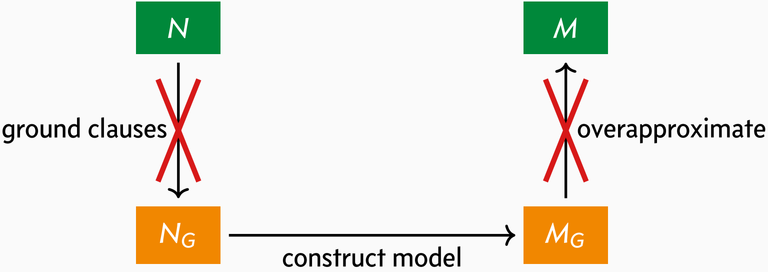
\end{tikzcd}$$

Completeness Proof

Goal

For every set N that is saturated, if N entails ⊥, then ⊥ ∈ N.



N

M

ground clauses

overapproximate

N_G

M_G

construct model

We add types to the nonground level.

Completeness Proof

Goal

For every set N that is saturated, if N entails $\bot$, then $\bot \in N$.

$$\begin{tikzcd}
N \arrow[d, "\text{ground welltyped clauses } \checkmark"'] & M \\
N_G \arrow[r, "\text{construct model}"'] & M_G \arrow[u, "\times" marking, "\text{overapproximate}"']
\end{tikzcd}$$

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Adapting the Inference Rules

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2. The renaming substitutions in the superposition rule are type preserving.
3. For each type, sufficiently many variables exist.
4. The variable-type environments in the superposition rule are compatible.
5. One additional side condition for the superposition rule.

Well-typedness Preservation

Theorem

Let C, D, and E be clauses and $\mathcal{V}_1$*,* $\mathcal{V}_2$*, and* $\mathcal{V}_3$ *be variable-type environments. If there exists an inference* superposition $\langle \mathcal{V}_1, D\rangle\ \langle \mathcal{V}_2, E\rangle\ \langle \mathcal{V}_3, C\rangle$*, then*

Well-typedness Preservation

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Let C, D, and E be clauses and $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ be variable-type environments. If there exists an inference superposition $\langle \mathcal{V}_1, D \rangle$ $\langle \mathcal{V}_2, E \rangle$ $\langle \mathcal{V}_3, C \rangle$*, then*

$$(\text{clause.is\_welltyped}\ \mathcal{F}\ \mathcal{V}_1\ D \wedge \text{clause.is\_welltyped}\ \mathcal{F}\ \mathcal{V}_2\ E) \longrightarrow \text{clause.is\_welltyped}\ \mathcal{F}\ \mathcal{V}_3\ C$$

Completeness Proof with Types

Theorem

For every set N that is saturated, if N entails $\bot$, then $\bot \in N$.

$$\begin{tikzcd}
N \arrow[d, swap, "\text{ground welltyped clauses } \checkmark"] & & M \\
N_G \arrow[rr, swap, "\text{construct model}"] & & M_G \arrow[u, swap, "\text{overapproximate } \checkmark"]
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Why Are Modular Proofs Useful?

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We are extending the approach to rank-1 polymorphism.

Conclusion

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Our goal is to obtain a verified executable superposition prover.