Proof-theoretic approach to description-logic

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Abstract

In recent work Baader has shown that a certain description logic with conjunction, existential quantification and with circular definitions has a polynomial time subsumption problem both under an interpretation of circular definitions as greatest fixpoints and under an interpretation as arbitrary fixpoints (introduced by Nebel). This was shown by translating definitions in the description logic ("TBoxes") into a labelled transition system and by reducing subsumption to a question of the existence of certain simulations. In the case of subsumption under the descriptive semantics a new kind of simulation, called synchronised simulation, had to be introduced. In this paper, we also give polynomial-time decision procedures for these logics; this time by devising sound and complete proof systems for them and demonstrating that proof search is polynomial for these systems. We then use the proof-theoretic method to study the hitherto unknown complexity of description logic with universal quantification, conjunction, and GCI axioms. Finally, we extend the proof-theoretic method to negation and thus obtain a decision procedure for the description logic ALC with fixpoints. This last section is only sketched.

1 Introduction

In [1] Baader has shown that a certain description logic with conjunction, existential quantification and with circular definitions has a polynomial time subsumption problem both under an interpretation of circular definitions as greatest fixpoints and under an interpretation as arbitrary fixpoints (deemed "descriptive semantics" by B. Nebel [6]).

This was shown by translating definitions in the description logic ("TBoxes") into a labelled transition system and by reducing subsumption to a question of the existence of certain simulations. In the case of subsumption under the descriptive semantics a new kind of simulation, called synchronised simulation, had to be introduced.

In this paper, we also give polynomial-time decision procedures for these logics; this time by devising sound and complete proof systems for them and demonstrating that proof search is polynomial time in these systems. While no genuinely new result, with the exception of the results in Section 7, is obtained we think that the paper is of value for at least the following reasons: the decision procedures provide a beautiful and new application of Gentzen-style proof theory; our proofs are simpler and less ad hoc than the original ones; our method could be more easily extensible to other systems. Of course, all these points are arguable.

2 Preliminaries

A *signature* is a pair $\Sigma = (\mathcal{R}, \mathcal{P})$ where $\mathcal{P}$ is a finite set of propositional letters and $\mathcal{R}$ is a finite set of relation symbols. Given a signature Σ the set of formulas is defined by the following grammar.

$$\phi, \psi ::= X \mid P \mid \phi \sqcap \psi \mid \exists r.\phi$$

Here X ranges over a set of propositional variables, P ranges over the set $\mathcal{P}$, and r ranges over the set $\mathcal{R}$.

A TBox is a list of equations of the form $X = \phi_X$ where a variable occurs at most once as a left hand side of an equation and the formulas on the right hand side only involve variables appearing as some left hand side. The formula ϕ_X may involve X, though, which amounts to a circular definition.

A formula *over a TBox* T is a formula containing no other variables than those defined in T.

$$\begin{array}{rcl} \mathcal{I}(X) & = & \mathcal{I}(\phi_X) \\ \mathcal{I}(\phi \sqcap \psi) & = & \mathcal{I}(\phi) \cap \mathcal{I}(\psi) \\ \mathcal{I}(\exists r.\phi) & = & \{x \mid \exists y {\in} \mathcal{I}(\phi). \mathcal{I}(r)(x,y)\} \end{array}$$

We remark that most authors define an interpretation as a mapping from *variables* and *letters* to subsets of $\mathcal{D}_{\mathcal{I}}$ which is then *extended* to all formulas by taking the above requirements as definitions. This definition is equivalent to the one given here.

Until further notice let T be fixed TBox. Greek letters ϕ, ψ, etc. will range over formulas over T. We say that ψ subsumes ϕ under the descriptive semantics if $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for all interpretations of the TBox. We write $\phi \sqsubseteq_{\mathrm{desc}} \psi$ to denote that situation. Below in Section 6 we will consider a coarser notion of subsumption called greatest fixpoint semantics.

We say that a formula ϕ is a subformula of a formula ψ if ϕ occurs literally as a subterm of ψ. We say that ϕ is a subformula of a TBox T if it equals one of its variables or is a subformula of one of its right-hand sides.

The following example is from [1]. We take $\mathcal{P} = \{\mathsf{Node}\}$ and $\mathcal{R} = \{\mathsf{edge}\}$. An interpretation $\mathcal{I}$ of this signature corresponds to a graph with node set $\mathcal{I}(\mathsf{Node})$ and egde relation $\mathcal{I}(\mathsf{edge})$. Now consider the circular definition $\mathsf{Inode} = \mathsf{Node} \sqcap \exists \mathsf{edge}.\mathsf{Inode}$. It asserts that from any Inode there emanates an edge leading to another Inode and moreover that if a Node x is linked to an Inode then x is an Inode itself. Of course, this does not uniquely determine the concept Inode. We may put $\mathcal{I}(\mathsf{Inode}) = \emptyset$ or else let $\mathcal{I}(\mathsf{Inode})$ be the set of nodes from which there issues an infinite path or else the points on some isolated circle.

While the interpretation of the circularly "defined" concept is not unique it makes sense to ask what can be deduced from the knowledge that a node is an Inode, in other words to study the subsumption problem. For example, if x is an Inode then there is a path of length three starting from x and leading to another Inode: we have $\mathsf{Inode} \sqsubseteq_{\mathrm{desc}} \exists \mathsf{edge}.\exists \mathsf{edge}.\exists \mathsf{edge}.\mathsf{Inode}$.

Below in Section 6 we introduce greatest fixpoint semantics which assigns a unique meaning to circularly defined concepts.

3 Proof system

We will now show that descriptive subsumption is decidable in polynomial time in the size of the TBox by devising a sound and complete Gentzen-style proof system.

The proof system operates on sequents of the form $\phi \sqsubseteq \psi$. Here the symbol $\sqsubseteq$ is a syntactic separator.

The proof rules of the system are as follows.

$$\frac{\phi \sqsubseteq \rho}{\phi \sqcap \psi \sqsubseteq \rho} \qquad (\textsc{AndL1})$$

$$\frac{\psi \sqsubseteq \rho}{\phi \sqcap \psi \sqsubseteq \rho} \qquad (\textsc{AndL2})$$

$$\frac{\phi \sqsubseteq \psi \qquad \phi \sqsubseteq \rho}{\phi \sqsubseteq \psi \sqcap \rho} \qquad (\textsc{AndR})$$

$$\frac{\phi \sqsubseteq \psi}{\exists r.\phi \sqsubseteq \exists r.\psi} \qquad (\textsc{Ex})$$

$$\frac{\phi_X \sqsubseteq \psi}{X \sqsubseteq \psi} \qquad (\textsc{DefL})$$

$$\frac{\psi \sqsubseteq \phi_X}{\psi \sqsubseteq X} \qquad (\textsc{DefR})$$

$$\phi \sqsubseteq \phi \qquad (\textsc{Ax})$$

The sequents above the horizontal line of a proof rule are called *antecedents* whereas the sequent below the horizontal line is called *conclusion*. The intuition is that if all antecedents of a rule have been derived then its conclusion may also be derived. Alternatively, a proof rule can be used backwards. The task of proving the conclusion of a rule can be reduced (by "invoking " the rule) to the task of providing proofs for each of the antecedents.

An *inductive proof* of a sequent $\phi \sqsubseteq \psi$ is a finite tree whose nodes are labelled with sequents, whose root is labelled with $\phi \sqsubseteq \psi$, whose terminal nodes (leaves) are labelled with axioms (instances of rule Ax) and whose internal nodes are always labelled with the conclusion of a proof rule whose antecedent(s) are the labellings of the children. In other words an inductive proof is what one would usually just call a "proof". We use the notation $\phi \sqsubseteq_{<\infty} \psi$ to indicate that the judgment $\phi \sqsubseteq \psi$ has an inductive proof.

Example 3.1 *Suppose that $X = P \sqcap \exists r.X$. Here is an inductive proof of $X \sqsubseteq \exists r.P \sqcap P$ thus establishing $X \sqsubseteq_{<\infty} \exists r.P \sqcap P$.*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \sqsubseteq P}\,\textsc{Ax}}{P \sqcap \exists r.X \sqsubseteq P}\,\textsc{AndL1}}{X \sqsubseteq P}\,\textsc{DefL} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \sqsubseteq P}\,\textsc{Ax}}{P \sqcap \exists r.X \sqsubseteq P}\,\textsc{AndL1}}{X \sqsubseteq P}\,\textsc{DefL}}{\exists r.X \sqsubseteq \exists r.P}\,\textsc{Ex}}{P \sqcap \exists r.X \sqsubseteq \exists r.P}\,\textsc{AndL2}}{X \sqsubseteq \exists r.P}\,\textsc{DefL}}{X \sqsubseteq \exists r.P \sqcap P}\,\textsc{AndR}$$

Lemma 3.2 (Cut elimination) *If $\phi \sqsubseteq_{<\infty} \psi$ and $\psi \sqsubseteq_{<\infty} \rho$ then $\phi \sqsubseteq_{<\infty} \rho$.*

Proof. The proof is entirely standard. For the sake of completeness we give an outline here.

Let $\mathcal{D}_1, \mathcal{D}_2$ denote the two derivations in question. We proceed by induction on both these derivations. If $\mathcal{D}_1$ ends with AndL1, AndL2, DefL, Ax the claim follows directly by applying he induction hypothesis to the immediate subderivation of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Suppose for example that $\phi = \phi_1 \sqcap \phi_2$ and that the last rule of $\mathcal{D}_1$ is ANDL1. The induction hypothesis then yields $\phi_1 \sqsubseteq_{<\infty} \rho$ and hence $\phi_1 \sqcap \phi_2 \sqsubseteq_{<\infty} \rho$ by ANDL1.

We proceed analogously if $\mathcal{D}_2$ ends with ANDR, DEFR, AX. In all other cases we can apply the induction hypothesis to the immediate subderivations of both $\mathcal{D}_1$ and $\mathcal{D}_2$ yielding the desired result directly. For example, if $\mathcal{D}_1$ ends with EX and none of the previous cases applies then $\mathcal{D}_2$ must end in EX, too. E.g., ANDL1 would be impossible. In this case, the induction hypothesis applies to both immediate subderivations and we conclude with EX. □

4 Deciding descriptive subsumption

Theorem 4.1 *$\phi \sqsubseteq_{\mathrm{desc}} \psi$ iff $\phi \sqsubseteq_{<\infty} \psi$, i.e., iff the judgment $\phi \sqsubseteq \psi$ has an inductive proof.*

Proof. If $\phi \sqsubseteq_{<\infty} \psi$ then clearly $\phi \sqsubseteq_{\mathrm{desc}} \psi$ by rule induction. All the proof rules are obviously sound with respect to the descriptive semantics.

For the converse we assume for simplicity that ϕ, ψ are subformulas of T (otherwise extend the TBox with two definitions $X = \phi, Y = \psi$ where X, Y are fresh) and construct an interpretation $\mathcal{D}_\mathcal{I}$ as follows. The domain $\mathcal{D}_\mathcal{I}$ is the set of subformulas of T. A propositional letter is interpreted by

$$\mathcal{I}(P) = \{\phi \mid \phi \sqsubseteq_{<\infty} P\}$$

A relation symbol r is interpreted by

$$\mathcal{I}(r)(\phi, \psi) \iff \phi \sqsubseteq_{<\infty} \exists r.\psi$$

Formulas, then, are interpreted by

$$\mathcal{I}(\phi) = \{\psi \mid \psi \sqsubseteq_{<\infty} \phi\}$$

Let us check that this is indeed an interpretation. We begin with the case $\phi = \phi_1 \sqcap \phi_2$. If $\psi \sqsubseteq_{<\infty} \phi_1 \sqcap \phi_2$ then we also have $\psi \sqsubseteq_{<\infty} \phi_i$ for $i = 1, 2$ by Lemma 3.2. Conversely, if $\psi \sqsubseteq_{<\infty} \phi_i$ for $i = 1, 2$ then $\psi \sqsubseteq_{<\infty} \phi_1 \sqcap \phi_2$ by ANDR.

The case $\phi = X$ is analogous.

Finally, consider the case where $\phi = \exists r.\phi_1$. Suppose $\psi \in \mathcal{I}(\exists r.\phi_1)$, i.e., $\psi \sqsubseteq_{<\infty} \exists r.\phi_1$. By definition, we then have $\mathcal{I}(r)(\psi, \phi_1)$ and of course $\phi_1 \in \mathcal{I}(\phi_1)$ by AX. Conversely, suppose that $\mathcal{I}(r)(\psi, \pi)$, i.e., $\psi \sqsubseteq_{<\infty} \exists r.\pi$ and $\pi \sqsubseteq_{<\infty} \phi_1$. We then have $\psi \sqsubseteq_{<\infty} \exists r.\phi_1$ by EX and Lemma 3.2.

Now, if $\phi \sqsubseteq_{\mathrm{desc}} \psi$ then in particular $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ by the definition of $\sqsubseteq_{\mathrm{desc}}$, so since $\phi \sqsubseteq_{<\infty} \phi$ by AX, we obtain $\phi \sqsubseteq_{<\infty} \psi$ as required. □

Theorem 4.2 *The relation $\sqsubseteq_{\mathrm{desc}}$ is decidable in polynomial time (in the size of the TBox and the two formulas involved).*

Proof. By the previous theorem it suffices to give a polynomial time decision procedure for the relation $\sqsubseteq_{<\infty}$. Suppose that we are given two formulas ϕ, ψ over the TBox. We may assume that both ϕ and ψ are actually subformulas of the TBox. All the proof rules have the subformula property in the sense that if the conclusion of an instance of any rule consists of subformulas of the TBox then the formulas in its premises are also subformulas of the TBox.

Thus it is clear that any proof of $\phi \sqsubseteq_{<\infty} \psi$ will only involve subformulas of the TBox. The number of these formulas is however polynomial (actually linear) in $|T|$.

We can thus compute a value table for $\sqsubseteq_{<\infty}$ restricted to these formulas in polynomial time by bottom-up iteration or dynamic programming.

More precisely, we set up an array A that contains a boolean entry $A[\phi, \psi]$ for any pair of subformulas ϕ, ψ of the TBox.

This array has size $O(n^2)$ when n is the size of the TBox. Initially, all the entries are set to "false". We now do the following repeatedly: for every instance of a proof rule with premises $\sigma_i \sqsubseteq \tau_i, i = 1, \ldots, \ell$ and conclusion $\phi \sqsubseteq \psi$ (Notice $\ell \in \{0, 1, 2\}$) with the property that $A[\sigma_i, \tau_i]$ equals "true" for $i = 1, \ldots, \ell$, set $A[\phi, \psi]$ equal to "true". Notice that the condition is vacuously true for rule AX so that $A[\phi, \phi]$ will be "true" after the first stage.

One stops as soon as this procedure stabilises, i.e., no more entries can be set to "true" that are not "true" already.

At this point one has $\phi \sqsubseteq_{<\infty} \psi$ iff $A[\phi, \psi]$ equals "true". The number of iterations required is $O(n^2)$. □

This performance of this procedure can be improved by filling the array in a demand-driven fashion by searching for a proof in a top-down fashion with possible backtracking in the case of rules ANDL1, ANDL2. All partial results obtained in the course of this search will of course be tabulated in the array and looked up rather than being recomputed when needed again. We remark that the described decision procedure is an instance of the general framework described by McAllister and Ganzinger [4].

5 Descriptive semantics with GCI axioms

We will now consider the problem where the TBox T in addition to fixpoint equations contains arbitrary assumptions of the form $\phi \sqsubseteq \psi$ where ϕ, ψ are formulas. Such assumptions are called *general concept inclusion* (GCI) axioms. The subsumption problem for this extension has been studied in [3] who gives the following instructive example of a GCI in the context of medical terminologies.

$$\mathsf{ulcer} \sqcap \exists \mathsf{has\_loc}.\mathsf{stomach} \sqsubseteq \exists \mathsf{has\_loc}.(\mathsf{lining} \sqcap \exists \mathsf{is\_part\_ofstomach})$$

This GCI asserts the fact that an ulcer located in the stomach will be located in the lining of the stomach. (The author is unable to assess the truth of this fact.)

An interpretation of a TBox with axioms is an interpretation of the TBox in the descriptive sense which in addition validates all the axioms, i.e., one has $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ whenever the axiom $\phi \sqsubseteq \psi$ is contained in the TBox.

Subsumption with respect to a TBox with axioms is defined as before, i.e., $\phi \sqsubseteq_{\mathrm{desc}} \psi$ if $\mathcal{I}(\phi) \sqsubseteq \mathcal{I}(\psi)$ for all interpretations of the TBox with axioms.

We claim that even in the presence of axioms this relation is decidable in polynomial time thus re-establishing the main result of [3]. To this end we augment our proof system by the following rule:

$$\frac{\rho \sqsubseteq \phi \qquad \psi \sqsubseteq \theta}{\rho \sqsubseteq \theta} \qquad (\text{CONCEPT})$$

whenever $\phi \sqsubseteq \psi \in \mathcal{T}$

Lemma 3.2 continues to hold for if one of the participating derivations has CONCEPT as its last inference then the induction hypothesis may be applied to one of the two subderivations of CONCEPT and the other derivation. One then concludes with an instance of CONCEPT. Of course, CONCEPT is sound.

It remains to show that the interpretation defined by $\mathcal{I}(\phi) = \{\psi \mid \psi \sqsubseteq_{<\infty} \phi\}$ indeed validates all the axioms of the TBox. If $\phi \sqsubseteq \psi$ is an axiom and $\alpha \in \mathcal{I}(\phi)$ then $\alpha \sqsubseteq_{<\infty} \phi$ by definition, so $\alpha \sqsubseteq_{<\infty} \psi$ by CONCEPT and AX.

Finally, the method of iteratively tabulating all derivable judgments of the form $\phi \sqsubseteq_{<\infty} \psi$ continues to work in the presence of rule CONCEPT so that we have shown PTIME decidability of descriptive subsumption with axioms.

Brandt (loc. cit.) also considers role inclusion axioms of the form $r \subseteq s$ with the understanding that $\mathcal{I}(r) \subseteq \mathcal{I}(s)$ whenever such an axiom is present. These can effectively be subsumed (sic!) under subsumption axioms by introducing an axiom $\exists r.\phi \sqsubseteq \exists s.\phi$ whenever ϕ is a subformula of the TBox. Of course, any interpretation that validates a role inclusion axiom $r \subseteq s$ will validate these new axioms. Conversely, if a judgement $\phi \sqsubseteq \psi$ follows from these new axioms then it will be validated by the "universal interpretation" $\mathcal{I}(\phi) = \{\rho \mid \rho \sqsubseteq_{<\infty} \phi\}$ and $\mathcal{I}(r) = \{(\phi, \psi) \mid \phi \sqsubseteq_{<\infty} \exists r.\psi\}$. However, this latter formula satisfies the role inclusion $r \subseteq s$ by rule CONCEPT.

6 Greatest fixpoint semantics

Let us revert to TBoxes with, possibly circular, definitions only. An interpretation of such a TBox under the *greatest fixpoint semantics* is an interpretation $\mathcal{I}$ which has the further property that whenever $\mathcal{J}$ is a function mapping formulas over the TBox to subsets of $\mathcal{D}_\mathcal{I}$ in such a way that

$$\begin{aligned} \mathcal{J}(P) &= \mathcal{I}(P) && (1)\\ \mathcal{J}(X) &\subseteq \mathcal{J}(\phi_X) && (2)\\ \mathcal{J}(\phi \sqcap \psi) &= \mathcal{J}(\phi) \cap \mathcal{J}(\psi) && (3)\\ \mathcal{J}(\exists r.\phi) &= \{x \mid \exists y {\in} \mathcal{J}(\phi).\mathcal{I}(r)(x,y)\} && (4) \end{aligned}$$

then $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all ϕ.

We remark that such an interpretation $\mathcal{I}$ is uniquely determined *by* and may be explicitly computed *from* its restriction to propositional letters by Knaster-Tarski's fixpoint theorem as the union of all functions $\mathcal{J}$ satisfying the above.

For example, in the example from Section 2 the greatest fixpoint semantics of lnode is the second of the possibilities given there, namely the set of all nodes from which there issues an infinite path.

We say that ψ subsumes ϕ under the greatest fixpoint semantics, written $\phi \sqsubseteq_{\mathrm{gfp}} \psi$, if $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for all interpretations $\mathcal{I}$ under the greatest fixpoint semantics.

Clearly, $\sqsubseteq_{\mathrm{gfp}}$ is a coarser relation than $\sqsubseteq_{\mathrm{desc}}$. For example, if the TBox contains the definitions $X = P \sqcap \exists r.X$ and $Y = P \sqcap \exists r.Y$ then $X \sqsubseteq_{\mathrm{gfp}} Y$ holds, while $X \sqsubseteq_{\mathrm{desc}} Y$ does not hold.

Indeed, the attempt of constructing a proof of $X \sqsubseteq Y$ in the above proof system leads to an infinite regress: one of the subgoals generated will be $X \sqsubseteq Y$ again. It is thus tempting to use a co-inductive interpretation of the proof rules, i.e., to ask for a possibly infinite proof tree.

Unfortunately, infinite proofs are not sound for greatest fixpoint subsumption. For example, if $\phi_X = X$ then we have an infinite proof of $X \sqsubseteq P$ for any P. This, however, is clearly unsound since the greatest fixpoint interpretation of such X is $\mathcal{I}(X) = \mathcal{D}_\mathcal{I}$.

Infinite proofs *are* sound if all definitions in the TBox are guarded in the sense that any self-referential occurrence of a variable in the right-hand side occurs in the scope of an $\exists r.$ operator. Alternatively, we may impose certain "liveness" conditions on infinite proofs that appropriately restrict the iterated usage of DEFL. Both kinds of restrictions are somewhat unwieldy to formalise which is why we prefer an indexed formulation of a proof system for greatest fixpoint subsumption.

More precisely, we define a family of relations $\phi \sqsubseteq_n \psi$ for $n \in \mathbb{N}$ as follows.

1. $\phi \sqsubseteq_0 \psi$ for all ϕ, ψ. (START)

2. $\phi \sqsubseteq_n \phi$ for all ϕ and all $n \in \mathbb{N}$.

3. The relations $\sqsubseteq_n$ are closed under the rules ANDL1, ANDL2, ANDR, DEFL, EX above.

4. If $\phi \sqsubseteq_n \phi_X$ then $\phi \sqsubseteq_{n+1} X$. (DEFR)

Let us write $\phi \sqsubseteq_\infty \psi$ to mean that $\phi \sqsubseteq_n \psi$ *for all* $n \in \mathbb{N}$.

Notice that if $\phi_X = X$ then $X \sqsubseteq_\infty P$ does *not* hold, in fact we do not even have $X \sqsubseteq_1 P$. Notice also that $\phi \sqsubseteq_m \psi$ implies $\phi \sqsubseteq_n \psi$ whenever $m \geq n$.

To see, why indeed all $n \in \mathbf{N}$ are needed, consider the TBox $X = P \sqcap \exists r.X, Y = P \sqcap \exists r.(P \sqcap \exists r.P)$. We have $Y \sqsubseteq_3 X$, but not $Y \sqsubseteq_4 X$.

We also remark that if $\phi \sqsubseteq_\infty \psi$ then by König's Lemma there exists a single infinite proof of $\phi \sqsubseteq \psi$ out of which proofs of $\phi \sqsubseteq_n \psi$ for arbitrary n arise by replacing appropriate subproofs by instances of START.

Establishing cut elimination (transitivity of $\sqsubseteq_\infty$) directly is somewhat awkward which is why we take an alternative route to soundness as follows.

Definition 6.1 *We write $\phi \Vdash \psi$ to denote that the judgement $\phi \sqsubseteq \psi$ can be derived with rules* AX, ANDL1, ANDL2, DEFL.

The following is direct.

Lemma 6.2 *If $\phi \Vdash \psi$ and $\psi \sqsubseteq_n \rho$ then $\phi \sqsubseteq_n \rho$.*

Lemma 6.3 (Generation) *Suppose that $n > 0$.*

1. *$\theta \sqsubseteq_n P$ iff $\theta \Vdash P$,*

2. *$\theta \sqsubseteq_n \psi_1 \sqcap \psi_2$ iff $\theta \sqsubseteq_n \psi_1$ and $\theta \sqsubseteq_n \psi_2$.*

3. *$\theta \sqsubseteq_n \exists r.\psi$ iff $\theta \Vdash \exists r.\rho$ and $\rho \sqsubseteq_n \psi$ for some subformula ρ of either the TBox or of $\theta, \exists r.\psi$.*

4. *$\theta \sqsubseteq_{n+1} X$ iff $\theta \sqsubseteq_n \phi_X$*

Proof. The "if" directions are immediate using appropriate proof rules and Lemma 6.2. For the "only if" directions we proceed by induction on derivations. The idea is that the only rules applicable are the ones defining $\Vdash$ or else the rule which decomposes the formula on the right hand side. Consider for example Case 3. If the last rule used was EX then we are done with $\rho = \psi$. Otherwise, the last rule used was one of AX, ANDL1,2, DEFL and we can apply the induction hypothesis and conclude with that same rule. $\square$

The following is clear.

Lemma 6.4 *Let $\mathcal{I}$ be an interpretation. If $\phi \Vdash \psi$ then $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$.*

Theorem 6.5 (Soundness) *If $\phi \sqsubseteq_\infty \psi$ then $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for any interpretation under the greatest fixpoint semantics.*

Proof. We want to show that for all θ, ϕ and interpretation $\mathcal{I}$ (under the greatest fixpoint semantics) we have

$$\theta \sqsubseteq_\infty \phi \Longrightarrow \mathcal{I}(\theta) \subseteq \mathcal{I}(\phi)$$

Equivalently, we may try to prove that for each ϕ and interpretation $\mathcal{I}$ one has

$$\bigcup_{\theta \sqsubseteq_\infty \phi} \mathcal{I}(\theta) \subseteq \mathcal{I}(\phi)$$

Writing $\mathcal{J}(\phi) := \bigcup_{\theta \sqsubseteq_\infty \phi} \mathcal{I}(\theta)$ for the left hand side this would follow if we can prove that $\mathcal{J}$ satisfies (dis)equations (1)-(4).

This, however, is a direct consequence of Lemma 6.3. Consider for example the case of Equation 3. If $x \in \mathcal{J}(\exists r.\phi)$ then by definition of $\mathcal{J}$ there exists a formula θ such that $\theta \sqsubseteq_\infty \exists r.\phi$ and $x \in \mathcal{I}(\theta)$. By Lemma 6.3 this means that we can find ρ such that $\theta \Vdash \exists r.\rho$ and $\rho \sqsubseteq_\infty \phi$. Strictly speaking, Lemma 6.3 may furnish a different ρ for each n, for a global one it suffices to take one ρ (of the finitely many possible) that occurs for infinitely many n. By Lemma 6.4 we now have $x \in \mathcal{I}(\exists r.\rho)$ so there exists $y \in \mathcal{I}(\rho)$ with $\mathcal{I}(r)(x, y)$. $\square$

Theorem 6.6 (Completeness) *If ϕ subsumes ψ under the greatest fixpoint semantics then $\psi \sqsubseteq_\infty \phi$.*

Proof. We use a similar interpretation as in the proof of Theorem 4.1 with $\sqsubseteq_{<\infty}$ replaced by $\sqsubseteq_\infty$ or $\Vdash$. More precisely,

$$\begin{aligned} \mathcal{D}_\mathcal{I} &= \text{subformulas of the TBox} \\ \mathcal{I}(r)(\phi, \psi) &\Leftrightarrow \phi \Vdash \exists r.\psi \\ \mathcal{I}(\phi) &= \{\psi \mid \psi \sqsubseteq_\infty \phi\} \end{aligned}$$

The fact that this interpretation satisfies the (1)-(4) is a direct consequence of Lemma 6.3. Consider the case where $\phi = \exists r.\phi_1$. If $\psi \in \mathcal{I}(\exists r.\phi_1)$ then $\psi \sqsubseteq_\infty \exists r.\phi_1$, so by Lemma 6.3 there exists ρ such that $\psi \Vdash \exists r.\rho$ and $\rho \sqsubseteq_\infty \phi_1$.

This means that $\mathcal{I}(r)(\psi, \exists r.\rho)$ and $\rho \in \mathcal{I}(\phi_1)$ as required. Conversely, if $\psi \Vdash \exists r.\rho$ and $\rho \sqsubseteq_\infty \phi_1$ then $\psi \sqsubseteq_\infty \exists r.\phi_1$ by EX and Lemma 6.2.

The other cases are analogous. It remains to show that $\mathcal{I}$ is indeed the greatest interpretation. To see this, assume that $\mathcal{J}$ is a function from formulas to subsets of $\mathcal{D}_\mathcal{I}$ that also satisfies (1)-(4) and coincides with $\mathcal{I}$ on propositional letters and relation symbols. We need to show that $\mathcal{J}(\psi) \subseteq \mathcal{I}(\psi)$ for all ψ. In other words, if $\phi \in \mathcal{J}(\psi)$ then $\phi \sqsubseteq_n \psi$ for all $n \in \mathbb{N}$. We show this by course of values induction on n and a subsidary induction on ψ.

If $\psi = P$ this is clear by assumption on $\mathcal{J}$

If $\psi = \psi_1 \sqcap \psi_2$ then $\phi \in \mathcal{J}(\psi_i)$ and we may inductively assume that $\phi \sqsubseteq_n \psi_i$ so $\phi \sqsubseteq_n \psi$ by ANDR.

If $\psi = \exists r.\psi_1$ then there exists a formula $\rho \in \mathcal{J}(\psi_1)$ with $\phi \Vdash \exists r.\rho$. The subordinate induction hypothesis applied to ψ_1 gives $\rho \sqsubseteq_n \psi_1$ and thus $\exists r.\rho \sqsubseteq_n \exists r.\psi_1$ by rule EX and thus $\rho \sqsubseteq_n \psi$ by Lemma 6.2.

If, finally, $\psi = X$ we distinguish two cases. If $n = 0$ then $\phi \sqsubseteq_n \psi$ by START. Otherwise, $n = n' + 1$ and $\psi \in \mathcal{J}(\phi_X)$ since $\mathcal{J}(X) \subseteq \mathcal{J}(\phi_X)$. The induction hypothesis gives $\phi \sqsubseteq_{n'} \phi_X$ and we obtain $\phi \sqsubseteq_n X$ by DEFR. □

Theorem 6.7 *The relation $\sqsubseteq_g$ (subsumption under the greatest fixpoint semantics) is decidable in polynomial time.*

Proof. By the previous result it suffices to decide the relation $\sqsubseteq_\infty$. As already mentioned, one has $\sqsubseteq_{n+1} \subseteq \sqsubseteq_n$, so one can compute these relations by iteration. More precisely we maintain two tables of polynomial size to hold the relations $\sqsubseteq_n$ and $\sqsubseteq_{n-1}$. Initially, we put $n = 0$ and set $\sqsubseteq_n$ to be the total relation in view of rule START. If relation $\sqsubseteq_{n-1}$ has already been computed we compute $\sqsubseteq_n$ from it by iteration as in the proof of Theorem 4.2 and so forth. As soon as we have reached an n_0 for which $\sqsubseteq_{n_0} = \sqsubseteq_{n_0+1}$ we may stop since for such n_0 one has $\sqsubseteq_\infty = \sqsubseteq_{n_0}$. But if $\sqsubseteq_{n+1} \subsetneq \sqsubseteq_n$ then $\sqsubseteq_{n+1}$ must contain at least one pair less than $\sqsubseteq_n$ which implies that n_0 exists and is polynomial in $|T|$ and the goal. This implies a polynomial runtime of the decision procedure. □

7 Universal quantification and GCI

After seeing an earlier version of this paper Baader asked the author about the complexity of the subsumption problem for a description logic that has a universal quantifier ($\forall r$) dual to the existential we already saw, conjunction ($\sqcap$) as before, and general concept inclusion (GCI) axioms. In particular, he suggested that this problem might lie in PSPACE. We will now show how the proof-theoretic method naturally leads to an EXPTIME algorithm for this subsumption problem and at the same time allows one to easily establish EXPTIME hardness. This is a new result although in the meantime Baader, Brandt, and Lutz simultaneously found an alternative proof of EXPTIME hardness using a reduction to Datalog [2]).

Signatures are defined as before as pairs $\Sigma = (\mathcal{R}, \mathcal{P})$. Formulas are now given by the grammar

$$\phi, \psi ::= P \mid \phi \sqcap \psi \mid \forall r.\phi$$

Variables are not needed in this case.

A TBox is a list of GCI axioms of the form $\phi \sqsubseteq \psi$ as in Section 4. Semantics is defined as in that Section with the additional requirement that

$$\mathcal{I}(\forall r.\phi) = \{x \mid \forall y{\in}\mathcal{D}_\mathcal{I}.\mathcal{I}(r)(x,y) \to y{\in}\mathcal{I}(\phi)\}$$

Subsumption $\phi \sqsubseteq_{\mathrm{desc}} \psi$ with respect to a TBox is defined as before as $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for all interpretations of the TBox. Our proof system now operates on sequents of the form $\Gamma \sqsubseteq \phi$ where Γ is a finite set of formulas.

The intended interpretation of such a sequent is $\phi_1 \sqcap \cdots \sqcap \phi_n \sqsubseteq_{\mathrm{desc}} \phi$ when $\Gamma = \{\phi_1, \ldots, \phi_n\}$. In this situation we write $\Gamma \sqsubseteq_{\mathrm{desc}} \phi$. We write Γ, ϕ for $\Gamma \cup \{\phi\}$. If $\Gamma = \{\phi_1, \ldots, \phi_n\}$ then we write $\forall r.\Gamma$ for $\{\forall r.\phi_1, \ldots, \forall r.\phi_n\}$. The proof rules are as follows.

$$\frac{\Gamma, \phi, \psi \sqsubseteq \rho}{\Gamma, \phi \sqcap \psi \sqsubseteq \rho} \quad (\text{ANDL})$$

$$\frac{\Gamma \sqsubseteq \phi \qquad \Gamma \sqsubseteq \psi}{\Gamma \sqsubseteq \phi \sqcap \psi} \quad (\text{ANDR})$$

$$\frac{\Delta \sqsubseteq \rho}{\Gamma, \forall r.\Delta \sqsubseteq \forall r \rho} \quad (\text{FORALL})$$

$$\frac{\phi \in \Gamma}{\Gamma \sqsubseteq \phi} \quad (\text{AX})$$

$$\frac{\Gamma \sqsubseteq \lambda \qquad \Gamma, \rho \sqsubseteq \phi}{\Gamma \sqsubseteq \phi} \quad (\text{CONCEPT})$$

where of course CONCEPT is applicable only when the GCI axiom $\lambda \sqsubseteq \rho$ is contained in the TBox. An intuitive reason as to why judgements of the form $\phi \sqsubseteq \psi$ are no longer sufficient stems from the fact that, unlike existential quantification, the universal quantification commutes with conjunction. Thus in particular, the following judgement should be provable: $(\forall r.\phi) \sqcap (\forall r.\psi) \sqsubseteq \forall r.\phi \sqcap \psi$. The proof is straightforward by rules AX, FORALL, ANDL, but there would be no proof if we had ANDL1 and ANDL2 instead of ANDL as before.

Let us write $\Gamma \sqsubseteq_{<\infty} \phi$ to indicate that the sequent $\Gamma \sqsubseteq \phi$ is provable in the sense that it has an inductive proof.

The following two are direct by induction on derivations.

Proposition 7.1 (Soundness) *If $\Gamma \sqsubseteq_{<\infty} \phi$ then $\Gamma \sqsubseteq_{\mathrm{desc}} \phi$.*

Lemma 7.2 (Weakening) *If $\Gamma \sqsubseteq_{<\infty} \phi$ then $\Gamma, \rho \sqsubseteq_{<\infty} \phi$ is provable by a proof of the same depth.*

Lemma 7.3 (Cut) *If $\Gamma \sqsubseteq_{<\infty} \phi$ and $\Gamma, \phi \sqsubseteq_{<\infty} \psi$ then also $\Gamma \sqsubseteq_{<\infty} \psi$.*

Proof. By induction on derivations. If the last rule of the first derivation is CONCEPT then we have shorter derivations of $\Gamma \sqsubseteq \lambda$ and $\Gamma, \rho \sqsubseteq \phi$. The induction hypothesis yields $\Gamma, \rho \sqsubseteq_{<\infty} \psi$ and we obtain $\Gamma \sqsubseteq_{<\infty} \psi$ by CONCEPT. If, on the other hand, the last rule of the second derivation was CONCEPT then we have shorter derivations of $\Gamma, \phi \sqsubseteq \lambda$ and $\Gamma, \phi, \rho \sqsubseteq \psi$. The induction hypothesis yields $\Gamma \sqsubseteq_{<\infty} \lambda$ and $\Gamma, \rho \sqsubseteq_{<\infty} \psi$. The result follows with rule CONCEPT. The other cases are similar. □

Theorem 7.4 (Completeness) *If* $\Gamma \sqsubseteq_{\mathrm{desc}} \phi$ *then* $\Gamma \sqsubseteq_{<\infty} \phi$.

Proof. As usual, we assume that all formulas in Γ, ϕ are mentioned in the TBox.

Let us write $\mathcal{F}$ for the set of subformulas of the TBox. We construct an interpretation $\mathcal{I}$ over the domain $\mathcal{D}_{\mathcal{I}} = \mathfrak{P}(\mathcal{F})$ containing sets of subformulas of the TBox.

Propositional letters are interpreted by $\mathcal{I}(P) = \{\Gamma \mid \Gamma \sqsubseteq_{<\infty} \phi\}$. Roles are interpreted by $\mathcal{I}(r)(\Gamma, \Delta)$ iff Δ consists of all $\phi \in \mathcal{T}$ for which $\Gamma \sqsubseteq_{<\infty} \forall r.\phi$. We extend the interpretation to all formulas by taking the required equations as an inductive definition.

Sublemma: If $\phi \in \mathcal{F}$ then $\mathcal{I}(\phi) = \{\Gamma \mid \Gamma \sqsubseteq_{<\infty} \phi\}$.

Proof of Sublemma: By induction on ϕ. If ϕ is a letter then this is true by definition. If $\phi = \phi_1 \sqcap \phi_2$ and $\Gamma \in \mathcal{I}(\phi) = \mathcal{I}(\phi_1) \cap \mathcal{I}(\phi_2)$ then by the induction hypothesis $\Gamma \sqsubseteq_{<\infty} \phi_1$ and $\Gamma \sqsubseteq_{<\infty} \phi_2$, so $\Gamma \sqsubseteq_{<\infty} \phi_1 \sqcap \phi_2$ by ANDR. Conversely, if $\Gamma \sqsubseteq_{<\infty} \phi_1 \sqcap \phi_2$ then also $\Gamma \sqsubseteq_{<\infty} \phi_i$ by ANDL and the Cut Lemma. If $\phi = \forall r.\phi_1$ and $\Gamma \in \mathcal{I}(\forall r.\phi)$ then let Δ be the (unique) set with $\mathcal{I}(r)(\Gamma, \Delta)$. We obtain $\Delta \in \mathcal{I}(\phi_1)$ so, by induction, $\Delta \sqsubseteq_{<\infty} \phi_1$. Rule ALL yields $\forall r.\Delta \sqsubseteq_{<\infty} \forall r.\phi$ and $\Gamma \sqsubseteq_{<\infty} \phi$ by the Cut Lemma and the defining property of Δ. Conversely, if $\Gamma \sqsubseteq_{<\infty} \forall r.\phi_1$ then $\phi_1 \in \Delta$ and $\Delta \sqsubseteq_{<\infty} \phi_1$ by AX. By induction $\Delta \in \mathcal{I}(\phi_1)$ and so $\Gamma \in \mathcal{I}(\phi)$. □

It remains to check that this interpretation indeed satisfies all GCI axioms: If $\lambda \sqsubseteq \rho$ is such an axiom and $\Gamma \in \mathcal{I}(\lambda)$ then $\Gamma \sqsubseteq_{<\infty} \lambda$ by the sublemma. On the other hand, $\Gamma, \rho \sqsubseteq_{<\infty} \rho$ by AX, so $\Gamma \sqsubseteq_{<\infty} \rho$ by CONCEPT and $\Gamma \in \mathcal{I}(\rho)$ by the sublemma.

Now, again by the sublemma and AX we have $\Gamma \in \bigcap_{\gamma \in \Gamma} \mathcal{I}(\gamma)$, so $\Gamma \in \mathcal{I}(\phi)$ and, finally, $\Gamma \sqsubseteq_{<\infty} \phi$ as required. □

Theorem 7.5 *There is an EXPTIME algorithm for deciding subsumption in description logic with* $\forall, \sqcap$ *and GCI axioms.*

Proof. The sound and complete proof system given above enjoys the subformula property. Thus there are only exponentially many possible sequents occurring in a proof. By dynamic programming one can systematically determine the provable ones among these. Alternatively, one may note that top down proof search can be performed by an alternating PSPACE machine. Either way, the EXPTIME bound follows. □

Conversely, deciding the relation $\sqsubseteq_{<\infty}$ and hence by soundness and completeness the relation $\sqsubseteq_{\mathrm{desc}}$ is EXPTIME hard.

We will show this by reduction to pushdown games which we describe here in a suitable version: Let Σ be a finite alphabet and Q be a finite set (of states) containing a special accepting state q_A. A configuration is a pair wq where $w \in \Sigma^*$ and $q \in Q$. A rule is an expression of the form $a_1, a_2 \to a_3$ where a_1, a_2, a_3 are configurations.

The pushdown game is played between two players I and II on configurations. In configuration a Player I chooses a rule $a_1, a_2 \to a_3$ such that $a = wa_3$ for some $w \in \Sigma^*$. Thereafter, Player II chooses a number $i \in \{1, 2\}$ and the subsequent configuration wa_i is reached. Player I wins once the configuration ϵq_A is reached. In all other cases Player II wins. It is EXPTIME-hard to decide whether Player I can win from a given configuration [8].

We remark that we can equivalently view a pushdown game as a Horn Theory whose propositional letters are the configurations and whose axioms are the Horn formulas of the form $wa_1, wa_2 \to wa_3$ where $a_1, a_2 \to a_3$ is a rule and $w \in \Sigma^*$ is arbitrary. Clearly, Player I wins the game iff q_A is provable in this Horn Theory. The pushdown games thus consitute an exponentially more concise schematic presentation of certain Horn Theories.

Suppose we are given such a pushdown game. We construct a TBox as follows: The propositional letters are the states of the game whereas the roles are the letters in Σ. For $w \in \Sigma^*$ we write $\forall w.\phi$ to abbreviate $\forall r_1 \dots \forall r_k.\phi$ when $w = r_1 \dots r_k$. A configuration wq is represented as the formula $\ulcorner wq \urcorner := \forall w.q$. A rule $a_1, a_2 \to a_3$ is represented by the GCI axiom $\ulcorner a_1 \urcorner \sqcap \ulcorner a_2 \urcorner \sqsubseteq \ulcorner a_3 \urcorner$.

Now q_A is subsumed by $\ulcorner a \urcorner$ if and only if Player I wins from a. To see this we first notice that the moves of a winning play from a immediately give rise to a proof of $q_A \sqsubseteq_{<\infty} \ulcorner a \urcorner$. Just mimick the moves of Player I by the appropriate choice of instances of CONCEPTS. For the converse, we argue semantically and consider an interpretation on the domain of pairs (w, a) where $w \in \mathcal{R}^*$ and a a configuration. The transition relation is given by $\mathcal{I}(r)((w, a), (wr, a))$ and the interpretation is generated by the following interpretation of the letters:

$$\mathcal{I}(q) = \{(w, a) \mid \text{Player I can force play from } wq \text{ to } a.\}$$

Now, by induction on the definition of $\ulcorner \_ \urcorner$ one obtains

$\mathcal{I}(\ulcorner a_1 \urcorner) = \{(w, a) \mid$ Player I can force play from wa_1 to $a.\}$

It follows that all GCI axioms are validated by this interpretation. Indeed, if $(w, a) \in \mathcal{I}(\ulcorner a_1 \urcorner \sqcap \ulcorner a_2 \urcorner)$ then Player I can force the play to a from both wa_1 and wa_2. In this case, he also wins from wa_3 by choosing the rule $a_1, a_2 \rightarrow a_3$.

Now, if $q_A \sqsubseteq_{<\infty} \ulcorner a \urcorner$ then, since $(\epsilon, q_A) \in \mathcal{I}(q_A)$ we have $(\epsilon, q_A) \in \mathcal{I}(\ulcorner a \urcorner)$ so that Player I wins from a.

Theorem 7.6 *Deciding subsumption in description logic with $\forall, \sqcap$ and GCI axioms is EXPTIME hard.*

8 Extension with negation (sketch)

We extend formulas with negation (and hence disjunction and universal quantification which then become definable) by adding a new clause $\phi ::= \dots |\neg\phi| \dots$ to the grammar for formulas.

The semantic clause for negation is

$$\mathcal{I}(\neg\phi) \quad = \quad \mathcal{D}_{\mathcal{I}} \setminus \mathcal{I}(\phi) \tag{5}$$

Furthermore, we require that every variable in ϕ_X appears under an even number of negation signs so that greatest fixpoints actually exist. We are in this section only interested in greatest fixpoint semantics, the descriptive semantics being actually easier and analogous.

In order to capture subsumption syntactically we introduce another proof system which this time operates on sequents of the form $\Gamma \sqsubseteq \Delta$ where Γ, Δ are (possibly empty) sets of formulas over the TBox.

In the sequel we use capital Greek letters to denote sets of formulas over the TBox, we write Γ, Δ for $\Gamma \cup \Delta$ and Γ, ϕ for $\Gamma \cup \{\phi\}$.

We have the following proof rules.

$$\frac{}{\Gamma \sqsubseteq_0 \Delta} \quad (\text{SSTART})$$

$$\frac{\Gamma \cap \Delta \neq \emptyset}{\Gamma \sqsubseteq_n \Delta} \quad (\text{SAX})$$

$$\frac{\Gamma \sqsubseteq_n \Delta, \phi}{\Gamma, \neg\phi \sqsubseteq_n \Delta} \quad (\text{SNEGL})$$

$$\frac{\Gamma, \phi \sqsubseteq_n \Delta}{\Gamma \sqsubseteq_n \Delta, \neg\phi} \quad (\text{SNEGR})$$

$$\frac{\Gamma, \phi, \psi \sqsubseteq_n \Delta}{\Gamma, \phi \sqcap \psi \sqsubseteq_n \Delta} \quad (\text{SANDL})$$

$$\frac{\Gamma \sqsubseteq_n \Delta, \phi \qquad \Gamma \sqsubseteq_n \Delta, \psi}{\Gamma \sqsubseteq_n \Delta, \phi \sqcap \psi} \quad (\text{SANDR})$$

$$\frac{\phi \sqsubseteq_n \Psi}{\Gamma, \exists r.\phi \sqsubseteq_n \exists r.\Psi, \Delta} \quad (\text{SEX})$$

$$\frac{\Gamma, \phi_X \sqsubseteq_n \Delta}{\Gamma, X \sqsubseteq_n \Delta} \quad (\text{SDEFL})$$

$$\frac{\Gamma \sqsubseteq_n \Delta, \phi_X}{\Gamma \sqsubseteq_{n+1} \Delta, X} \quad (\text{SDEFR})$$

Given an interpretation (in the greatest fixpoint sense) we define

$$\begin{aligned} \mathcal{I}^{\sqcup}(\Gamma) &= \bigcup_{\phi \in \Gamma} \mathcal{I}(\phi) \\ \mathcal{I}^{\sqcap}(\Gamma) &= \bigcap_{\phi \in \Gamma} \mathcal{I}(\phi) \end{aligned}$$

Here the empty intersection is understood to be the set $\mathcal{D}_{\mathcal{I}}$ whereas the empty union is the empty intersection.

Theorem 8.1 $\phi \sqsubseteq_{\text{gfp}} \psi$ *if and only if* $\phi \sqsubseteq_n \psi$ *is provable for all* $n \in \mathbb{N}$.

The proof of this result will appear in the full paper; it follows essentially the proof of Theorems 6.5 and 6.6 above with the notable complication that the completeness proof now requires the use of maximally consistent sets of formulas.

Dynamic programming as before now directly yields an EXPTIME algorithm for subsumption (notice that the number of sequents is now exponential) matching the known lower bound.

9 Conclusion and related work

We have given sound and complete proof systems for description logic with circular definitions both under the descriptive semantics and under a greatest fixpoint interpretation. We also have considered the case of arbitrary axioms. The main tool was Gentzen-style proof theory and dynamic programming to obtain algorithms from proof systems.

While such proof systems were known for various modal logics they seem to be new for the extension with circular definitions. Also the indexed formulation of a proof system for greatest fixpoints appears to be new.

The idea of systematically tabulating derivable judgements has also been used by Ganzinger and McAllister [4] to derive efficient algorithms and complexity bounds from logic programming formulations of solutions to problems.

The programme set out in this paper can be carried out for other description logics and various combinations of features. Superficial investigations show that in each case the known bounds are achieved. It remains to be seen to what extent further new results can be achieved in this way.

Of course, the description logic with greatest fixpoint semantics is a subsystem of Kozen's μ-calculus [5] and thus logics and decision procedures for the latter can in principle also used in our context. However, cut-free proof systems for μ-calculus are not available[1] and thus do not give rise to decision procedure. Indeed, all known decision procedures for μ-calculus rely on automata-theoretic methods. Moreover, all these methods have exponential complexity due to the EXPTIME hardness of μ-calculus. We have attempted to extend the method of indexed, infinite proofs from Section 6 to μ-calculus, but not yet to complete satisfaction.

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