

The most important problem in the area of Bounded Arithmetic is whether the hierarchy of theories

$$PV = T_2^0 \subseteq S_2^1 \subseteq T_2^1 \subseteq S_2^2 \dots$$

is proper or not, and to determine what conservativity relations hold between these theories. These questions are strongly related to difficult open questions in computational complexity theory.

The questions are more accessible in the relativized setting, where the theories are extended by an uninterpreted predicate constant  $\alpha$ . In this setting, the relationship between  $T_2^i(\alpha)$  and  $S_2^{i+1}(\alpha)$  is well-understood: the latter is conservative over the former w.r.t.  $\forall\Sigma_{i+1}^b(\alpha)$ -sentences, but not w.r.t. any larger class of sentences.

The relationship between  $S_2^i(\alpha)$  and  $T_2^i(\alpha)$  is less clear, it is conjectured that the former is not even conservative w.r.t.  $\forall\Sigma_1^b$ -sentences over the latter, for every  $i \geq 1$ . The best known result in general here is non-conservativity w.r.t.  $\forall\Sigma_i^b$ -sentences.

An important step towards the resolution of these questions is a computational or combinatorial characterization of the  $\Sigma_1^b$ -consequences of the theories as some classes of NP search problems. An oracle separation of these classes of search problems would then yield the non-conservativity of the theories w.r.t.  $\Sigma_1^b$ -formulas.

The paper under review gives new characterizations of the  $\Sigma_1^b$ -consequences of the theories  $T_2^2$  and  $T_2^3$ . These characterizations are of three different kinds:

Generalizing the well-known characterization of the NP search problems definable in  $T_2^1$  by polynomial local search problems [1], a class of colored polynomial local search problems is introduced and shown to coincide with the search problems definable in  $T_2^2$ .

The  $\Sigma_1^b$ -consequences of  $T_2^2$  are then characterized by a reflection principle for propositional resolution proofs, and this is generalized to  $T_2^3$  by considering a generalization of resolution that allows cuts on arbitrary conjunctions.

Finally, the  $\Sigma_1^b$ -consequences of  $T_2^2$  are given a rather technical characterization in terms of the provable termination of certain recursive program schemes.

## References

- [1] S. R. Buss and J. Krajíček, An application of Boolean complexity to separation problems in bounded arithmetic, Proc. London Math. Soc. (3) **69** (1994), no. 1, 1–21. MR1272418 (96b:03074)