The Axiom of Infinity in Zermelo Set Theory*

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Abstract

In this note we show that the deductive extension of Zermelo set theory depends on the formulation of the axiom of infinity. In particular, we prove that the original version given by Zermelo is not sufficient to prove the existence of the set ω of von Neumann natural numbers.

Introduction

Whereas in ZF the special formulation of the axiom of infinity is not important, it makes a difference in absence of the axiom of replacement. Although this fact is not difficult to prove, it seems to be widely unknown.

Let Z_0 denote Zermelo set theory without the axiom of infinity, i.e. the theory given by the axioms of extensionality, separation, pairing, union, powerset and regularity in their standard first-order formulation, like e.g. given in [1].

We use standard set theoretic terminology and notation. Classes are used as meta-expressions as usual, and for a class $\mathcal{C}$ and formula ϕ, $\mathcal{C} \models \phi$ is an abbreviation for the relativization of φ to $\mathcal{C}$.

We shall use the well-known fact that bounded formulae (i.e. such that involve only bounded quantifiers $\forall x \in y$, $\exists x \in y$) are absolute w.r.t. transitive classes; this is to say that for a bounded formula $\varphi(x)$ and a transitive class $\mathcal{C}$, $\varphi(a) \leftrightarrow (\mathcal{C} \models \varphi(a))$ is provable for every $a \in \mathcal{C}$. A proof can be found in any textbook on set theory, e.g. in [1]. In particular, the axioms of extensionality and regularity are equivalent to bounded formulae, and are therefore valid in every transitive class.

The axioms of infinity

Let ζ denote the set $\{\, \emptyset\, , \{\emptyset\}\, , \{\{\emptyset\}\}\, , \ldots\}$, i.e. the smallest set containing the empty set and closed under singletons. Let ω be the set of finite von Neumann ordinals, and let HF be the set of hereditarily finite sets. Although the existence of any of these sets

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cannot be proved in Z_0, they can be defined as classes in such a way that they have all desired properties.

The first form of the axiom of infinity we shall consider is simply "there is an infinite set", i.e. one in which every element of ω can be injected:

$$\exists y \, \forall x \, (x \in \omega \rightarrow x \preceq y)$$

We shall see that this axiom, which is denoted (inf_1) in the sequel, does not imply the existence of any of the sets ζ, ω or HF in Z_0. (inf_2) is the original version of the axiom of infinity given by Zermelo in [4]:

$$\exists y \, (\emptyset \in y \wedge \forall x \, (x \in y \rightarrow \{x\} \in y))$$

(inf_3) is the usual axiom of infinity as it is found e.g. in [1]:

$$\exists y \, (\emptyset \in y \wedge \forall x \, (x \in y \rightarrow s(x) \in y))$$

where $s(x)$ denotes $x \cup \{x\}$. The following stronger axiom (inf_4) is also sometimes found:

$$\exists y \, (\emptyset \in y \wedge \forall x_1, x_2 \, (x_1 \in y \wedge x_2 \in y \rightarrow x_1 \cup \{x_2\} \in y))$$

For $i = 1, \ldots, 4$, let Z_i denote the theory Z_0 together with the axiom (inf_i).

Now in Z_2 the existence of the set ζ is obviously provable, and the same holds for Z_3 and ω. Furthermore, in either case we can prove the scheme of induction for the respective set,

$$Z_2 \vdash \varphi(\emptyset) \wedge \forall x \, (\varphi(x) \rightarrow \varphi(\{x\})) \rightarrow \forall x \, (x \in \zeta \rightarrow \varphi(x))$$

and similarly for Z_3 with ζ replaced by ω. Likewise, we can prove the existence of HF in Z_4. Thus we can conclude that $Z_4 \vdash (\mathrm{inf}_i)$ for $i = 2, 3$ since $\emptyset \in HF$ and HF is closed under singletons and s.

Now we show that $Z_2 \vdash (\mathrm{inf}_1)$. Define a function r on ζ by

$$r(\emptyset) = \emptyset, \qquad r(\{x\}) = s(r(x)),$$

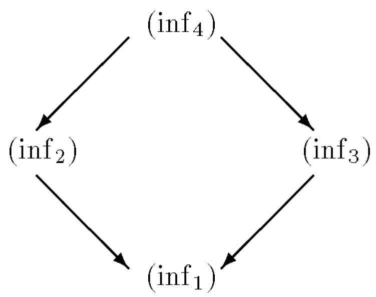
and for a given $n \in \omega$, define the function $f_n : n \rightarrow \zeta$ by

$$f_n = \{(x, y) \in n \times \zeta; \, x = r(y)\}$$

Then, since r is injective, f_n is an injective mapping from n to ζ.

Finally we have $Z_3 \vdash (\mathrm{inf}_1)$: For $n \in \omega$, the identity mapping $id_n := \{(i, i); i \in n\}$ is an injective mapping from n into ω.

So we have the following diamond of implications between the axioms of infinity w.r.t. Z_0:



We shall see in the following that none of the arrows can be reversed, and especially that (inf_2) and (inf_3) are incomparable, in contrast to the situation in ZF, where all four versions are equivalent.

The model constructions

By an *inner model* of a theory T, we mean a class $\mathcal{C}$ such that $T \vdash (\mathcal{C} \models \varphi)$ for every axiom φ of T. We shall now give two inner models $\mathcal{N} \models \mathrm{Z}_3$ and $\mathcal{Z} \models \mathrm{Z}_2$ where the sets ζ and ω resp. do not exist.

Theorem 1 *There is an inner model $\mathcal{N}$ of* Z_3 *such that*

$$\mathcal{N} \models \neg \exists x \; x = \zeta$$

Proof: First we describe the model $\mathcal{N}$ informally: Let

$$\begin{aligned} \mathcal{N}_0 &:= \omega \\ \mathcal{N}_{i+1} &:= \mathcal{P}(\mathcal{N}_i) \\ \mathcal{N} &:= \bigcup_{i<\omega} \mathcal{N}_i \end{aligned}$$

We shall prove in Z_3 that $\mathcal{N}$ is indeed a model of Z_3, and that $\zeta \notin \mathcal{N}$. The class $\mathcal{N}$ can be defined by the following formula:

$$\begin{aligned} x \in \mathcal{N} \quad :\leftrightarrow \quad & \exists f \,(\, \mathrm{fun}(f) \wedge \mathrm{dom}(f) \in \omega \wedge f(0) = \omega \\ & \wedge \forall i \in \omega \,(s(i) \in \mathrm{dom}(f) \rightarrow f(s(i)) = \mathcal{P}(f(i))) \wedge \exists i \in \mathrm{dom}(f) \; x \in f(i)) \end{aligned}$$

First we shall show that $\mathcal{N}$ is transitive: Let $a \in \mathcal{N}$, then by there is a function f as required by the definition and an $i \in \omega$ such that $a \in f(i)$. Then either $i = \emptyset$, then $a \in \omega$, and since ω is transitive, $a \subset \omega$, or $i = s(j)$ for some $j \in \omega$, then $a \in \mathcal{P}(f(j))$, and thus $a \subseteq f(j)$, and in either case $a \subseteq \mathcal{N}$.

Since $\mathcal{N}$ is a transitive class, $\mathcal{N}$ is a model of the axioms of extensionality and regularity.

Next, $\mathcal{N}$ is closed under pairs and unions, and since these notions can be defined by bounded formulae, they are absolute w.r.t. $\mathcal{N}$, so $\mathcal{N}$ is a model of the corresponding axioms.

To show that $\mathcal{N}$ is a model of the axiom scheme of separation means that for every formula φ and every $a \in \mathcal{N}$ we have to prove

$$\exists y \in \mathcal{N}\, \forall x \in \mathcal{N}\ (x \in y \leftrightarrow x \in a \wedge \mathcal{N} \models \varphi(x))$$

But if $a \in \mathcal{N}_i$, then $\{x \in a; \mathcal{N} \models \varphi(x)\}$ is a subset of $\mathcal{N}_i$, and hence an element of $\mathcal{N}_{i+1}$, so we have found an $y \in \mathcal{N}$ that fulfills the requirement.

Also $\mathcal{N}$ is closed under powersets, and a similar argument using the absoluteness of subsets shows that the powerset axiom is true in $\mathcal{N}$. Finally, $\omega \in \mathcal{N}$, and since it is absolute, $\mathcal{N}$ is a model of (inf_3).

Now ζ is not an element of $\mathcal{N}$ since

$$\underbrace{\{\ldots\{}_{i+2}\emptyset\underbrace{\}\ldots\}}_{i+2} \in \mathcal{N}_{i+1} \setminus \mathcal{N}_i$$

and since ζ can be defined by a bounded formula, it is absolute and hence ζ does not exist in the model $\mathcal{N}$. $\square$

Since (inf_4) implies (inf_2), $\mathcal{N} \models \neg(\mathrm{inf}_4)$. Therefore neither of these two axioms is a theorem of Z_3. Finally, since $(\mathrm{inf}_3) \to (\mathrm{inf}_1)$ we have $\mathcal{N} \models (\mathrm{inf}_1)$ and thus (inf_1) does not imply the existence of the set ζ.

Theorem 2 *There is an inner model $\mathcal{Z}$ of Z_2 such that*

$$\mathcal{Z} \models \neg\exists x\ x = \omega$$

Proof: The model $\mathcal{Z}$ is defined completely analogous to the model $\mathcal{N}$: We let

$$\begin{aligned} \mathcal{Z}_0 &:= \zeta \\ \mathcal{Z}_{i+1} &:= \mathcal{P}(\mathcal{Z}_i) \\ \mathcal{Z} &:= \bigcup_{i<\omega} \mathcal{Z}_i \end{aligned}$$

Then $\mathcal{Z}$ is a model of Z_2, and $\omega \notin \mathcal{Z}$ by the same argument as above. The class $\mathcal{Z}$ can be defined by the formula

$$\begin{aligned} x \in \mathcal{Z} \quad :\leftrightarrow \quad & \exists f\,(\mathrm{fun}(f) \wedge \mathrm{dom}(f) \subset \zeta \wedge \mathrm{trans}(\mathrm{dom}(f)) \wedge f(0) = \zeta \\ & \wedge \forall i \in \zeta\,(\{i\} \in \mathrm{dom}(f) \to f(\{i\}) = \mathcal{P}(f(i))) \wedge \exists i \in \mathrm{dom}(f)\ x \in f(i)) \end{aligned}$$

Note that a transitive *proper* subset d of ζ must be a finite initial segment of ζ: if $\{x\} \in d$, then by transitivity $\{x\} \subseteq d$, and thus $x \in d$. Hence, in order to be a proper subset of ζ, d must be finite.

Since the definition of $\mathcal{Z}$ mirrors the definition of $\mathcal{N}$ we can prove the claims of the theorem just like in the proof of Theorem 1, using induction on ζ where induction on ω is used there. $\square$

Again, since $(\mathrm{inf}_4) \to (\mathrm{inf}_3)$, Z_2 does not prove that HF exists. Finally, since $(\mathrm{inf}_2) \to (\mathrm{inf}_1)$, ($\mathrm{inf}_1$) does not imply the existence of ω.

The axioms of Parlamento and Policriti

In [2, 3] two formulae $\varphi_i(a,b)$ $i = 1,2$ (of the lowest possible logical complexity) are defined such that $\exists x, y\, \varphi_i(x,y)$ is equivalent to (inf$_3$) w.r.t. ZF without the axiom of infinity. These are

$$\begin{aligned}
\varphi_1(a,b) \quad &:\leftrightarrow \quad a \neq b \wedge a \notin b \wedge b \notin a \\
&\wedge \forall x \in a\, (x \subseteq b) \wedge \forall x \in b\, (x \subseteq a) \wedge \forall x \in a\, (x \notin b) \\
&\wedge \forall x, y \in a\, \forall z, w \in b\, (z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y) \\
&\wedge \forall x, y \in b\, \forall z, w \in a\, (z \in x \wedge x \in w \wedge w \in y \rightarrow z \in y) \\
\varphi_2(a,b) \quad &:\leftrightarrow \quad a \neq \emptyset \wedge b \neq \emptyset \wedge a \notin b \wedge b \notin a \\
&\wedge \forall x \in a\, (x \subseteq b) \wedge \forall x \in b\, (x \subseteq a) \\
&\wedge \forall x \in a\, \forall y \in b\, (x \in y \vee y \in x)
\end{aligned}$$

We now want to investigate how these axioms fit into our framework. First we have that

$$\mathrm{Z}_0 \vdash \varphi_2(a,b) \rightarrow \varphi_1(a,b)$$

The proof of this fact given in [3] requires only extensionality and regularity.

In [2] it is proved (in ZF), that if $\varphi_1(a,b)$ holds, then either $\omega' \subseteq a$ and $\omega'' \subseteq b$ or vice versa, where

$$\omega' := \{f_i \,;\, i \in \omega\} \quad \text{and} \quad \omega'' = \{g_i \,;\, i \in \omega\}$$

with $f_0 := \emptyset$, $g_n := \{f_0, \ldots, f_n\}$ and $f_{n+1} := \{g_0, \ldots, g_n\}$. The proof given there requires — besides the axioms of Z_0 — some form of induction and can therefore be carried out in Z_2 or Z_3. On the other hand, $\varphi_2(\omega', \omega'')$ holds, and since these are easily definable subsets of HF, we have immediately:

$$\mathrm{Z}_4 \vdash \exists x, y\, \varphi_2(x,y)$$

We shall now show that $\exists x, y\, \varphi_1(x,y)$ is not provable in Z_2 or Z_3. Consider the models we have constructed for these theories. We shall see that the sets ω' and ω'' are not elements of $\mathcal{N}$ and $\mathcal{Z}$.

A closer inspection of the definitions of these sets shows that $f(i+1) \in \mathcal{N}_{2i+1} \setminus \mathcal{N}_{2i}$ and $g(i+1) \in \mathcal{N}_{2i+2} \setminus \mathcal{N}_{2i+1}$. Thus ω' and ω'' cannot be elements of $\mathcal{N}$. Likewise we have $f(i+2) \in \mathcal{Z}_{2i+2} \setminus \mathcal{Z}_{2i+1}$ and $g(i+1) \in \mathcal{Z}_{2i+1} \setminus \mathcal{Z}_{2i}$. Therefore they are also not elements of $\mathcal{Z}$.

Instead of directly constructing a model of $\mathrm{Z}_0 + \exists x, y\, \varphi_1(x,y)$, we shall state the following general observation that we made while proving this fact.

Let us call a set *n-transitive* if

$$\forall x \in a\, \forall y \in x\;\, y \in \bigcup_{i \leq n} \mathcal{P}^i(a)$$

Thus 0-transitive means transitive in the usual sense.

Theorem 3 *Let a be n-transitive for some $n \in \omega$, then $\mathcal{M}$ defined by*

$$\begin{aligned} \mathcal{M}_0 &:= a \\ \mathcal{M}_{i+1} &:= \mathcal{M}_i \cup \mathcal{P}(\mathcal{M}_i) \\ \mathcal{M} &:= \bigcup_{i<\omega} \mathcal{M}_i. \end{aligned}$$

is a model of Z_0.

Proof: $\mathcal{M}_n$ is transitive, for let $x \in \mathcal{M}_n$, then *either* $x \in a$, hence $x \subseteq \bigcup_{i \leq n} \mathcal{P}^i(a)$, and thus $x \subseteq \mathcal{M}_n$, *or* $x \in \mathcal{M}_{i+1} \setminus \mathcal{M}_i$ for some $i < n$, hence $x \in \mathcal{P}(\mathcal{M}_i)$ and so $x \subseteq \mathcal{M}_i$.

It follows immediately that $\mathcal{M}$ is transitive, and hence $\mathcal{M}$ is a model of extensionality and regularity.

The validity of the remaining axioms is proved similar to the corresponding parts in the proof of Theorem 1, with some extra care for the elements appearing in the layers $\mathcal{M}_i$ $(i < n)$ below $\mathcal{M}_n$. □

Now a model $\mathcal{M}$ of $\mathrm{Z}_0 + \exists x, y\ \varphi_1(x, y)$ where neither of the sets ω and ζ exists is constructed by taking $a = \omega'$ in the previous Theorem, since ω' is 2-transitive. Then $\omega'' \in \mathcal{M}_2$, and thus $\mathcal{M} \models \exists x, y\ \varphi_1(x, y)$. $\omega \notin \mathcal{M}$ and $\zeta \notin \mathcal{M}$ can be seen by the same method as $\omega' \notin \mathcal{N}$.

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