A Note on Eventually Complete Models of Type Theory and Quine's New Foundations

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Abstract

We define a property of models of simple type theory, viz. that of being *eventually complete*, and show that the existence of such models is equivalent to the consistency of Quine's New Foundations. Furthermore we show that the usual standard models are not eventually complete, although for models with an infinite domain of individuals, we lack examples of sentences witnessing this.

Introduction

By TST, the simple theory of types, we understand the following theory: the types are the natural numbers, and there are variables of every type, where the type of a variable is indicated by an upper index. The formulae are built up from atomic formulae of the form $x^i \in y^{i+1}$ and $x^i = y^i$, and the quantifiers can bind variables of every type.

The axioms of TST are *extensionality*

$$\forall x^i \left(x^i \in a^{i+1} \leftrightarrow x^i \in b^{i+1}\right) \to a^{i+1} = b^{i+1}$$

for every type i, and the *axiom schema of comprehension*

$$\exists y^{i+1}\, \forall x^i \left(x^i \in y^{i+1} \leftrightarrow \varphi(x^i)\right)$$

for every type i and every formula $\varphi(x^i)$ with free variable x^i in which the variable y^{i+1} does not occur free. A standard model of TST is one where for every i, the elements of type $i+1$ form the full powerset of the set of elements of type i, and $\in$ is the real elementhood relation.

If φ is a formula in the language of TST and $i \in \mathbb{N}$, then $\varphi^{(i)}$ denotes the formula that results from φ by adding i to every type index. Instead of $\varphi^{(1)}$ we also write φ^{+}.

The axiom schema of *typical ambiguity* (Amb for short) is $\varphi \leftrightarrow \varphi^{+}$ for every sentence φ. For a class Γ, $Amb(\Gamma)$ denotes the schema of typical ambiguity for all $\varphi \in \Gamma$. Note that the rule of proof "from φ conclude φ^{+}" is admissible in TST.

For a formula φ in the language of TST, let $\hat{\varphi}$ denote the formula in the first-order language of set theory that results from φ by erasing all type indices. A set-theoretic formula is called *stratified* if it is of the form $\hat{\varphi}$ for some φ. On the other hand, if η is a stratified set-theoretic formula, then we denote by $\check{\eta}$ its *canonical stratification*, i.e. the unique formula φ in the language of TST using the smallest possible types such that $\eta = \hat{\varphi}$.

Quine's New Foundations [4] is the first-order set theory NF whose axioms are the sentences $\hat{\varphi}$ for all axioms φ of TST. The consistency of NF is a long-standing open problem. By a theorem of Specker, it is equivalent to the consistency of $TST + Amb$. It is well-known (see e.g. [2]) that NF proves the existence of infinite sets, as well as the negation of the axiom of choice.

We now define the central notions of this note. For a model $\mathfrak{M}$ of TST and a sentence φ we say that $\mathfrak{M}$ *eventually satisfies* φ if there is an $n \in \mathbb{N}$ s.t. $\mathfrak{M} \models \varphi^{(i)}$ for every $i \geq n$. E.g. cardinality statements like "there are at least n objects of type i" are eventually satisfied in every model of TST.

We call $\mathfrak{M}$ *eventually complete* if for every sentence φ (in the language of TST), $\mathfrak{M}$ eventually satisfies φ or $\mathfrak{M}$ eventually satisfies $\neg\varphi$. Note that $\mathfrak{M}$ is eventually complete iff the set of sentences eventually satisfied in $\mathfrak{M}$ is complete, which justifies the — admittedly slightly awkward — terminology.

Finally we say that a sentence φ *oscillates* in a model $\mathfrak{M}$ if $\mathfrak{M} \models \varphi^{(i)}$ for infinitely many $i \in \mathbb{N}$ and $\mathfrak{M} \models \neg\varphi^{(j)}$ for infinitely many $j \in \mathbb{N}$. Hence an oscillating sentence in $\mathfrak{M}$ witnesses that $\mathfrak{M}$ is not eventually complete.

Using an ultrapower construction for models of TST, we shall show that the consistency of NF is equivalent to the existence of eventually complete models of TST. Furthermore we show that standard models of TST are not eventually complete, and we give examples of sentences oscillating in standard models with a finite domain of individuals.

The model construction

Let $\mathfrak{M} := \Big(\langle M_j\rangle_{j\geq 0}\,,\,\langle \in_j\rangle_{j\geq 0}\Big)$ be a structure for the language of TST, i.e. $\in_j\ \subseteq M_j \times M_{j+1}$.

For $i \in \mathbb{N}$ we define $\mathfrak{M}^{(i)}$ to be the structure $\Big(\langle M_j\rangle_{j\geq i}\,,\,\langle \in_j\rangle_{j\geq i}\Big)$. In other words, $\mathfrak{M}^{(i)}$ denotes the structure obtained from $\mathfrak{M}$ by forgetting the i lowest types. Then the following proposition is quite obvious:

Proposition 1. *For every sentence φ in the language of TST,*

$$\mathfrak{M}^{(i)} \models \varphi \quad \textit{iff} \quad \mathfrak{M} \models \varphi^{(i)}\ .$$

Let D be an ultrafilter on ω. We denote the ultraproduct $\prod_{i\in\omega} \mathfrak{M}^{(i)}/D$ by $\vec{\mathfrak{M}}^{\omega}/D$ and call it the *skew ultrapower* of $\mathfrak{M}$. An account of the ultraproduct construction for higher order models can be found in [1].

The skew ultrapower has the following property:

Proposition 2. *For every sentence φ in the language of TST,*

$$\vec{\mathfrak{M}}^{\omega}/D \models \varphi \quad \textit{iff} \quad \Big\{\, i\ ;\ \mathfrak{M} \models \varphi^{(i)} \,\Big\} \in D\ .$$

Proof. By Łoś' fundamental theorem on ultraproducts, $\vec{\mathfrak{M}}^{\omega}/D \models \varphi$ iff $\big\{\, i\ ;\ \mathfrak{M}^{(i)} \models \varphi \,\big\} \in D$, and thus Prop. 1 yields the claim. □

In particular, $\vec{\mathfrak{M}}^{\omega}/D \models TST$, since $\omega \in D$ for every ultrafilter D. If D is the principal ultrafilter generated by $\{i\}$, then $\vec{\mathfrak{M}}^{\omega}/D$ is isomorphic to $\mathfrak{M}^{(i)}$.

Applications

The following theorem connecting typical ambiguity to the consistency of first-order theories was essentially proved by Specker, although he did not formulate it in its full generality.

Theorem 3 (Specker). *Let Φ be a set of stratified first-order sentences. Then Φ is consistent iff $\check{\Phi} = \{\check{\varphi}\ ;\ \varphi \in \Phi\} + Amb$ is consistent.*

In particular, this shows that the consistency of NF is equivalent to the existence of a model of $TST+Amb$, since NF is axiomatized by a set of stratified sentences. For an outline of the proof see Thm. 2.3.1 and Lemma 2.3.2 in [2] or the original [5, 6].

We now relate typical ambiguity to eventual completeness.

Theorem 4. *If $\mathfrak{M} \models TST$ is eventually complete and D is a nonprincipal ultrafilter, then $\vec{\mathfrak{M}}^\omega/D \models TST + Amb$.*

Proof. It suffices to show that it is impossible that for some sentence φ, $\vec{\mathfrak{M}}^\omega/D \models \varphi$ and $\vec{\mathfrak{M}}^\omega/D \models \neg\varphi^+$. By Prop. 2, the first statement is equivalent to $F_1 := \{i\,;\, \mathfrak{M} \models \varphi^{(i)}\} \in D$, and the second one is equivalent to $F_2 := \{i\,;\, \mathfrak{M} \models \neg\varphi^{(i+1)}\} \in D$.

However, since $\mathfrak{M}$ is eventually complete, one of the sets F_1 and F_2 must be finite, and hence cannot be an element of the nonprincipal ultrafilter D. □

On the other hand, if a model of TST satisfies typical ambiguity, it is obviously eventually complete, hence we have:

Corollary 5. *NF is consistent iff there is an eventually complete model of TST.*

A slight refinement of the proof of the above Thm. 4 together with Specker's Theorem 3 yields the following:

Theorem 6. *For every stratified sentence φ, NF proves φ iff every eventually complete model of TST eventually satisfies $\check{\varphi}$.*

Proof. Suppose there is an eventually complete model $\mathfrak{M}$ of TST eventually satisfying $\neg\check{\varphi}$. If D is a nonprincipal ultrafilter, then $\vec{\mathfrak{M}}^\omega/D \models TST + Amb + \neg\check{\varphi}$, and by Specker's Theorem 3 we get a model of $NF + \neg\varphi$.

On the other hand let $NF + \neg\varphi$ be consistent, then again Thm. 3 yields a model of $TST + Amb + \neg\check{\varphi}$, and of course this model is eventually complete, and does not eventually satisfy $\check{\varphi}$. □

An application is e.g. the following: Let AC denote the axiom of choice in the following form: for every set a of pairwise disjoint nonempty sets, there is a choice set, i.e. a set whose intersection which each element of a has exactly one element. This can obviously be written as a stratified sentence. Then, since NF proves $\neg AC$, no model of TST satisfying $\check{AC}^{(i)}$ for every i is eventually complete. In particular, we have:

Corollary 7. *No standard model of TST is eventually complete.*

Another consequence is the following: Let $\mathfrak{M}$ be a standard model with M_0 finite, and let FIN be a sentence in the language of TST saying that there are only finitely many objects of type 0. Then $\mathfrak{M}$ cannot be eventually complete, since $\mathfrak{M} \models FIN^{(i)}$ for every $i \in \mathbb{N}$, whereas NF proves $\neg \hat{FIN}$.

Note that for such $\mathfrak{M}$, the skew ultrapower $\vec{\mathfrak{M}}^\omega/D$ has the property of satisfying $FIN^{(i)}$ for every i, as well as the statement "there are at least n elements of type i" for every n and every type i. This shows that the notion of finiteness cannot be expresses in TST in a completely satisfying way.

Let $\forall_4$ denote the prefix class $\forall^*\exists^*\forall^*\exists^*$. In [3] the equiconsistency of NF and $TST + Amb(\forall_4)$ was proved. Hence we could replace "eventually complete" by "eventually complete for $\forall_4$-sentences" throughout the above arguments, and where we assert the existence of oscillasting sentences, we can further conclude that oscillating sentences of this form must exist.

Oscillating sentences

For the standard models $\mathfrak{M}$ with a finite set M_0 of individuals, oscillating sentences can be explicitely constructed as follows:

Define the set of Frege-Russell natural numbers $\mathcal{N}_i$ of type $i+3$ as the set of equivalence classes of equipollent sets of type $i+1$. Note that $\mathcal{N}_i$ contains the natural numbers from 0 through $|M_i|$. On these, the relation $y = 2^x$ can be defined: y is the set of all sets of type $i+1$ that are equipollent to the powerset of some set in x.

Let S_i be the least subset of $\mathcal{N}_i$ such that $|M_0| \in S_i$ and whenever $x \in S_i$ and $2^x \in \mathcal{N}_i$, then $2^x \in S_i$, so that $S_i = \{\, |M_j| \,;\, j \leq i \,\}$. By formalizing this construction we can write down a sentence φ such that $\varphi^{(i)}$ expresses that the cardinality of S_i is even. Thus $\mathfrak{M} \models \varphi^{(i)}$ iff i is odd, so φ oscillates in $\mathfrak{M}$.

This construction can be modified in such a way that the numeral $|M_0|$ is not mentioned explicitely. This yields one sentence that oscillates in every standard model $\mathfrak{M}$ with M_0 finite. In fact, assuming GCH or the weaker hypothesis that the function $\kappa \mapsto 2^\kappa$ is injective on infinite cardinals, we can prove that this sentence oscillates in all standard models. We were not able to come up with an example of a sentence of which we can prove that it oscillates in any standard model with an infinite domain of individuals without using strong set-theoretic hypotheses.

Thus it is a challenge to even find a sentence oscillating in the standard model with $|M_0| = \aleph_0$, provably in ZFC. Another challenge is to find simpler oscillating sentences in the case M_0 finite, since the sentences constructed above are much more complex than $\forall_4$. One class of simple sentences is ruled out as examples by the following:

Observation: No sentence equivalent to a boolean combination of statements $|M_0| \equiv r \pmod{n}$ for some $r, n \in \mathbb{N}$ oscillates in any standard model of TST.

Proof. This is obvious for M_0 infinite, so we assume M_0 is finite. Let $v_i := |M_i|$ for $i \in \mathbb{N}$, hence $v_{i+1} = 2^{v_i}$. Then for each $n \geq 1$, the sequence $\langle v_i \bmod n; i \in \mathbb{N} \rangle$ is eventually constant, more precisely: for every $n \geq 1$ and $i \geq n$: $v_{i+1} \equiv v_i \pmod{n}$.

This is proved by induction on n as follows: let $n = 2^w m$ with $2 \nmid m$. Since $2^w \leq n \leq i \leq v_i = 2^{v_{i-1}}$, we have $w \leq v_{i-1}$, and hence $v_{i+1} \equiv v_i \pmod{n}$ is equivalent to

$$2^{v_i - w} \equiv 2^{v_{i-1} - w} \pmod{m} .$$

Now if $m = 1$, this is trivial. Otherwise we must have $m > 2$, hence the above congruence is equivalent to

$$\begin{aligned} & v_i - w \equiv v_{i-1} - w \pmod{\operatorname{ord}_m(2)} \\ \leftrightarrow \quad & v_i \equiv v_{i-1} \pmod{\operatorname{ord}_m(2)} \end{aligned}$$

and since $1 \leq \operatorname{ord}_m(2) < n$ and $(i-1) \geq \operatorname{ord}_m(2)$, this is an instance of the induction hypothesis. □

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