Elements Definable by Nonstandard Σ_n-Formulae in Models of Peano Arithmetic

Jan Johannsen
Universität Erlangen-Nürnberg
IMMD 1, Martensstraße 3, D-91058 Erlangen
Tel.: +49 9131 857928
email: johannsen@informatik.uni-erlangen.de

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Let $M \models PA$ be nonstandard, and I a proper cut in M. We assume a Gödel-numbering of syntax and semantics as in Chapter 9 of [2] and use the notation of this book, but unlike Kaye we do not distinguish between formulae and elements $a \in M$ satisfying $form(a)$. λ denotes the (code of the) empty sequence, and $a * b$ the sequence that results from appending the number b to the sequence a.

Throughout we assume that I is *closed*, i.e. if φ and ψ are formulae in I and x is a variable in I, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\exists x\, \varphi$ and $\forall x\, \varphi$ are also in I. In most places, we would only need a weaker condition, namely that I is closed under $\wedge$, $\vee$ and existential quantification for Σ_n-formulae, but for a natural Gödel-numbering the two notions coincide.

A *satisfaction class* on M is a subset $S \subseteq M \times M$ such that if $(\varphi, a) \in S$, then $M \models form(\varphi)$ and a is a sequence of elements of M whichs length is at least the (possibly nonstandard) number of free variables in φ, and the model M expanded by S satisfies the Tarskian truth conditions formulated in the language of PA with a binary relation symbol S, based on a truth definition for atomic formulas (cf. [2, Ch. 15] or [3]).

On every model M there exists the standard satisfaction class S_0, the set of pairs $(\varphi, \bar{a})$ where φ is a standard formula and $M \models \varphi(\bar{a})$.

For $n \geq 1$, we define the set $K_I^n(M, S)$ of those elements in M which are definable by (non-standard) Σ_n-formulae in I using the satisfaction class S by

$$K_I^n(M,S) := \Big\{ b \in M \ ;\ \exists \varphi \in I\ \exists a \in I$$
$$(M,S) \models form_{\Sigma_n}(\varphi) \wedge S(\varphi, a * b) \wedge \forall x\, S(\varphi, a * x) \rightarrow x = b \Big\}\ .$$

At first glance, it might seem superfluous to work with satisfaction classes when dealing with Σ_n-formulae only, since there is a definable satisfaction relation $Sat_{\Sigma_n}(\varphi, a)$ for such formulae. Nevertheless, when using this definition, every nonstandard Σ_n-formula gets a fixed value for each assignment, so we lose a possibility of variation. That such possibility exists shows the following

Proposition 1 *There is a countable model $M \models PA$, a $\varphi \in M$ such that $M \models form_{\Delta_0}(\varphi)$ and satisfaction classes S_1, S_2 on M such that $(M, S_1) \models S(\varphi, \lambda)$ and $(M, S_2) \models \neg S(\varphi, \lambda)$.*

Proof: Let M be such that there is a full, Δ_0-inductive satisfaction class S_1 on M. Let furthermore $\varphi :\equiv \bigwedge_{i<a} 0 = 0$ for some $\mathbb{N} < a \in M$. Then since S_1 is Δ_0-inductive, $(M, S_1) \models S(\varphi, \lambda)$.

On the other hand, since by fullness of S_1, M must be recursively saturated, the construction of [2, Thm. 15.6] yields another (so-called weakly-$\wedge$-pathological) satisfaction class S_2 with $(M, S_2) \models S(\neg\varphi, \lambda)$. □

A satisfaction class S is called $\Sigma_n(I)$*-full* if for every Σ_n-formula φ in I and every valuation $a \in I$ (i.e. a sequence of elements of I of suitable length) either $S(\varphi, a)$ or $S(\neg\varphi, a)$ holds in (M, S).

Theorem 2 *If S is $\Sigma_n(I)$-full, then $K_I^n(M, S) \prec_{\Sigma_n} M$.*

Proof: First it is obvious that $K_I^n(M, S)$ is a substructure of M if the conditions of the theorem are fulfilled.

It suffices to show the following: If $\bar{b} \in K_I^n(M, S)$ and $\varphi(x, \bar{y})$ is a Π_{n-1}-formula such that $M \models \exists x\, \varphi(x, \bar{b})$, then there is a $c \in K_I^n(M, S)$ such that $M \models \varphi(c, \bar{b})$. By induction, there is a unique $c \in M$ such that

$$M \models \varphi(c, \bar{b}) \wedge \forall x < c\, \neg\varphi(x, \bar{b}) \ .$$

We only have to show that $c \in K_I^n(M, S)$. Let the parameters $\bar{b} = b_1, \ldots, b_k$ be defined by nonstandard Σ_n-formula $\beta_1, \ldots, \beta_k$ in I with free variables $\bar{y}, x$ and a sequence of parameters $\bar{a} \in I$, i.e. for each $i \leq k$

$$(M, S) \models S(\beta_i, \bar{a} * b_i) \wedge \forall x\, S(\beta_i, \bar{a} * x) \rightarrow x = b_i \ .$$

By collection, $\forall x < z\, \neg\varphi(x, b)$ is equivalent in M to a Σ_{n-1}-formula $\psi(z, \bar{v})$. Now let

$$\eta :\equiv \exists\bar{v}\, \beta_1(\bar{y}, v_1) \wedge \ \ldots \ \wedge \beta_k(\bar{y}, v_k) \wedge \varphi(z, \bar{v}) \wedge \psi(z, \bar{v})$$

then since I is closed, η is a Σ_n-formula in I, and since furthermore S is $\Sigma_n(I)$-full, we have that

$$(M, S) \models S(\eta, \bar{a} * c) \wedge \forall x\, S(\eta, \bar{a} * x) \rightarrow x = c$$

by the properties of a satisfaction class and the fact that $k \in \mathbb{N}$ and φ and ψ are standard formulae. $\square$

From the Theorem we immediately get the following

Corollary 3 *If S is $\Sigma_n(I)$-full, then $K^n_I(M,S) \models I\Sigma_{n-1}$.*

Let Sat_{Σ_n} be a natural truth definition for Σ_n-formulae. Then we say that a satisfaction class S on M is $\Sigma_n(I)$*-compatible*, if for every Σ_n-formula in I, and every valuation $a \in I$

$$(M,S) \models S(\varphi,a) \leftrightarrow Sat_{\Sigma_n}(\varphi,a) \ .$$

Recall that the formula Sat_{Σ_n} is equivalent to a Σ_n-formula in PA.

Theorem 4 *If S is $\Sigma_n(I)$-full and $\Sigma_n(I)$-compatible and $I \subsetneq K^n_I(M,S)$, then $K^n_I(M,S) \not\models B\Sigma_n$.*

Proof: Let $b \in K^n_I(M,S)$, then there is a Σ_n-formula $\varphi \in I$ and a sequence of parameters $a \in I$ such that $(M,S) \models S(\varphi, a*b) \wedge \forall x\,(S(\varphi,a*x) \rightarrow x = b)$. Consider the standard formula

$$\eta(b,w) := \exists \varphi, a\,(w = \langle \varphi, a\rangle \wedge Sat_{\Sigma_n}(\varphi, a*b)) \ ,$$

which is equivalent to a Σ_n-formula in M, and let $c \in K^n_I(M,S) \setminus I$, which is non-empty by assumption. Since I is closed, the pair $\langle \varphi, a\rangle$ is in I, and thus

$$M \models \exists w\!<\!c\,\eta(b,w)$$

by S being $\Sigma_n(I)$-compatible. But this formula is Σ_n, hence by Thm. 2 we have

$$K^n_I(M,S) \models \forall b\!\leq\! c\,\exists w\!<\!c\,\eta(b,w)$$

since $b \in K^n_I(M,S)$ was arbitrary. Now suppose $K^n_I(M,S) \models B\Sigma_n$, then the last sentence would be equivalent in $K^n_I(M,S)$ to a Σ_n-formula, and hence by Thm. 2 again, M would also satisfy $\forall b\!\leq\! c\,\exists w\!<\!c\,\eta(b,w)$. On the other hand,

$$M \models \eta(b_1,w) \wedge \eta(b_2,w) \rightarrow b_1 = b_2$$

for suppose M satisfies $\eta(b_1,w)$ and $\eta(b_2,w)$ for $w = \langle \varphi, a\rangle$, then we would have by $\Sigma_n(I)$-compatibility $S(\varphi, a*b_1)$ and $S(\varphi, a*b_2)$ both hold in (M,S), hence $b_1 = b_2$.

But then $\eta(b,w)$ would define a $1-1$ map from $c+1$ to c in M, and so the pigeonhole principle in M (cf. [1]) would be violated. $\square$

Observe that we can easily find a model $M \models PA$, $I \subseteq_e M$ and a $\Sigma_n(I)$-full and $\Sigma_n(I)$-compatible satisfaction class S on M such that I and $K^n(M) = K^n_{\mathbb{N}}(M, S_0)$ are both properly contained in $K^n_I(M, S)$:

Let M be such that $K^n(M)$ is nonstandard, then $K^n(M)$ is not an initial segment of M, since a Σ_n-elementary initial segment of a model of PA satisfies $B\Sigma_{n+1}$. Let I be the initial segment generated by $K^n(M)$, i.e.

$$I := \{\, a \in M \,;\, \exists b \in K^n(M)\; a < b \,\} \ .$$

Then if $n \geq 2$, I is closed since $I \models I\Sigma_{n-1}$. In the case $n = 1$, replace I by the smallest initial segment containing $K^n(M)$ that is closed under exponentiation.

Define a satisfaction class

$$S := \{\, (\varphi, a) \,;\, \varphi \in I \text{ and } M \models Sat_{\Sigma_n}(\varphi, a) \,\} \ ,$$

which has the desired properties simply by definition.

Then obviously $K^n(M) \subsetneqq K^n_I(M, S)$, since $K^n(M) \subsetneqq I$. On the other hand, the above results imply that $I \subsetneqq K^n_I(M, S)$, since $K^n_I(M, S)$ cannot be an initial segment of M.

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References

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