Elements Definable by Nonstandard $\Sigma_n$-Formulae in Models of Peano Arithmetic

Jan Johannsen
Universität Erlangen-Nürnberg
IMMD 1, Martensstraße 3, D-91058 Erlangen
Tel.: +49 9131 857928
e-mail: johannsen@informatik.uni-erlangen.de

August 4, 1995

Let $M \models PA$ be nonstandard, and $I$ a proper cut in $M$. We assume a Gödel-numbering of syntax and semantics as in Chapter 9 of [2] and use the notation of this book, but unlike Kaye we do not distinguish between formulae and elements $a \in M$ satisfying $\text{form}(a)$. $\lambda$ denotes the (code of the) empty sequence, and $a \ast b$ the sequence that results from appending the number $b$ to the sequence $a$.

Throughout we assume that $I$ is closed, i.e. if $\varphi$ and $\psi$ are formulae in $I$ and $x$ is a variable in $I$, then $\varphi \land \psi$, $\varphi \lor \psi$, $\neg \varphi$, $\exists x \varphi$ and $\forall x \varphi$ are also in $I$. In most places, we would only need a weaker condition, namely that $I$ is closed under $\land$, $\lor$ and existential quantification for $\Sigma_n$-formulae, but for a natural Gödel-numbering the two notions coincide.

A satisfaction class on $M$ is a subset $S \subseteq M \times M$ such that if $(\varphi, a) \in S$, then $M \models \text{form}(\varphi)$ and $a$ is a sequence of elements of $M$ which length is at least the (possibly nonstandard) number of free variables in $\varphi$, and the model $M$ expanded by $S$ satisfies the Tarskian truth conditions formulated in the language of $PA$ with a binary relation symbol $S$, based on a truth definition for atomic formulas (cf. [2, Ch. 15] or [3]).

On every model $M$ there exists the standard satisfaction class $S_0$, the set of pairs $(\varphi, \bar{a})$ where $\varphi$ is a standard formula and $M \models \varphi(\bar{a})$.

For $n \geq 1$, we define the set $K_n^\varphi(M, S)$ of those elements in $M$ which are definable by (non-standard) $\Sigma_n$-formulae in $I$ using the satisfaction class $S$ by

\[ K_n^\varphi(M, S) := \left\{ b \in M \mid \exists \varphi \in I \exists a \in I \right. \]

\[ (M, S) \models \text{form}_{\Sigma_n}(\varphi) \land S(\varphi, a \ast b) \land \forall x S(\varphi, a \ast x) \rightarrow x = b \} . \]
At first glance, it might seem superfluous to work with satisfaction classes when dealing with $\Sigma_n$-formulae only, since there is a definable satisfaction relation $Sat_{\Sigma_n}(\varphi, a)$ for such formulae. Nevertheless, when using this definition, every nonstandard $\Sigma_n$-formula gets a fixed value for each assignment, so we lose a possibility of variation. That such possibility exists shows the following

**Proposition 1** There is a countable model $M \models PA$, a $\varphi \in M$ such that $M \models form_{\Delta_0}(\varphi)$ and satisfaction classes $S_1, S_2$ on $M$ such that $(M, S_1) \models S(\varphi, \lambda)$ and $(M, S_2) \models \neg S(\varphi, \lambda)$.

**Proof:** Let $M$ be such that there is a full, $\Delta_0$-inductive satisfaction class $S_1$ on $M$. Let furthermore $\varphi \equiv \bigwedge_{i < a} 0 = 0$ for some $N < a \in M$. Then since $S_1$ is $\Delta_0$-inductive, $(M, S_1) \models S(\varphi, \lambda)$.

On the other hand, since by fullness of $S_1$, $M$ must be recursively saturated, the construction of [2, Thm. 15.6] yields another (so-called weakly-$\land$-pathological) satisfaction class $S_2$ with $(M, S_2) \models S(\neg \varphi, \lambda)$. \qed

A satisfaction class $S$ is called $\Sigma_n(I)$-full if for every $\Sigma_n$-formula $\varphi$ in $I$ and every valuation $a \in I$ (i.e. a sequence of elements of $I$ of suitable length) either $S(\varphi, a)$ or $S(\neg \varphi, a)$ holds in $(M, S)$.

**Theorem 2** If $S$ is $\Sigma_n(I)$-full, then $K^*_S(M, S) \prec_{\Sigma_n} M$.

**Proof:** First it is obvious that $K^*_S(M, S)$ is a substructure of $M$ if the conditions of the theorem are fulfilled.

It suffices to show the following: If $\bar{b} \in K^*_S(M, S)$ and $\varphi(x, \bar{g})$ is a $\Pi_{n-1}$-formula such that $M \models \exists x \varphi(x, \bar{b})$, then there is a $c \in K^*_S(M, S)$ such that $M \models \varphi(c, \bar{b})$. By induction, there is a unique $c \in M$ such that

\[ M \models \varphi(c, \bar{b}) \land \forall x < c \neg \varphi(x, \bar{b}) . \]

We only have to show that $c \in K^*_S(M, S)$. Let the parameters $\bar{b} = b_1, \ldots, b_k$ be defined by nonstandard $\Sigma_n$-formula $\beta_1, \ldots, \beta_k$ in $I$ with free variables $\bar{g}, x$ and a sequence of parameters $\bar{a} \in I$, i.e. for each $i \leq k$

\[ (M, S) \models S(\beta_i, \bar{a} \ast b_i) \land \forall x . S(\beta_i, \bar{a} \ast x) \rightarrow x = b_i . \]

By collection, $\forall x < z \neg \varphi(x, \bar{b})$ is equivalent in $M$ to a $\Sigma_{n-1}$-formula $\psi(z, \bar{v})$. Now let

\[ \eta := \exists \beta_1(\bar{g}, v_1) \land \ldots \land \beta_k(\bar{g}, v_k) \land \varphi(z, \bar{v}) \land \psi(z, \bar{v}) \]

then since $I$ is closed, $\eta$ is a $\Sigma_n$-formula in $I$, and since furthermore $S$ is $\Sigma_n(I)$-full, we have that

\[ (M, S) \models S(\eta, \bar{a} \ast c) \land \forall x . S(\eta, \bar{a} \ast x) \rightarrow x = c . \]
by the properties of a satisfaction class and the fact that \( k \in \mathbb{N} \) and \( \varphi \) and \( \psi \) are standard formulae. \( \square \)

From the Theorem we immediately get the following

**Corollary 3** If \( S \) is \( \Sigma_n(I) \)-full, then \( K^n_I(M, S) \models I\Sigma_{n-1} \).

Let \( \text{Sat}_{\Sigma_n} \) be a natural truth definition for \( \Sigma_n \)-formulae. Then we say that a satisfaction class \( S \) on \( M \) is \( \Sigma_n(I) \)-compatible, if for every \( \Sigma_n \)-formula in \( I \), and every valuation \( a \in I \)

\[
(M, S) \models S(\varphi, a) \leftrightarrow \text{Sat}_{\Sigma_n}(\varphi, a)
\]

Recall that the formula \( \text{Sat}_{\Sigma_n} \) is equivalent to a \( \Sigma_n \)-formula in \( PA \).

**Theorem 4** If \( S \) is \( \Sigma_n(I) \)-full and \( \Sigma_n(I) \)-compatible and \( I \not\models K^n_I(M, S) \), then \( K^n_I(M, S) \not\models B\Sigma_n \).

**Proof:** Let \( b \in K^n_I(M, S) \), then there is a \( \Sigma_n \)-formula \( \varphi \in I \) and a sequence of parameters \( a \in I \) such that \((M, S) \models S(\varphi, a \ast b) \land \forall x (S(\varphi, a \ast x) \rightarrow x = b) \). Consider the standard formula

\[
\eta(b, w) := \exists \varphi, a \ (w = \langle \varphi, a \rangle \land \text{Sat}_{\Sigma_n}(\varphi, a \ast b))
\]

which is equivalent to a \( \Sigma_n \)-formula in \( M \), and let \( c \in K^n_I(M, S) \setminus I \), which is non-empty by assumption. Since \( I \) is closed, the pair \( \langle \varphi, a \rangle \) is in \( I \), and thus

\[
M \models \exists w < c \eta(b, w)
\]

by \( S \) being \( \Sigma_n(I) \)-compatible. But this formula is \( \Sigma_n \), hence by Thm 2 we have

\[
K^n_I(M, S) \models \forall b \leq c \exists w < c \eta(b, w)
\]

since \( b \in K^n_I(M, S) \) was arbitrary. Now suppose \( K^n_I(M, S) \models B\Sigma_n \), then the last sentence would be equivalent in \( K^n_I(M, S) \) to a \( \Sigma_n \)-formula, and hence by Thm 2 again, \( M \) would also satisfy \( \forall b \leq c \exists w < c \eta(b, w) \). On the other hand,

\[
M \models \eta(b_1, w) \land \eta(b_2, w) \rightarrow b_1 = b_2
\]

for suppose \( M \) satisfies \( \eta(b_1, w) \) and \( \eta(b_2, w) \) for \( w = \langle \varphi, a \rangle \), then we would have by \( \Sigma_n(I) \)-compatibility \( S(\varphi, a \ast b_1) \) and \( S(\varphi, a \ast b_2) \) both hold in \((M, S)\), hence \( b_1 = b_2 \).

But then \( \eta(b, w) \) would define a \( 1 - 1 \) map from \( c + 1 \) to \( c \) in \( M \), and so the pigeonhole principle in \( M \) (cf. [1]) would be violated. \( \Box \)
Observe that we can easily find a model $M \models PA$, $I \subseteq M$ and a $\Sigma_n(I)$-full and $\Sigma_n(I)$-compatible satisfaction class $S$ on $M$ such that $I$ and $K^n(M) = K^n_{\beta}(M, S_0)$ are both properly contained in $K^n_{\beta}(M, S)$:

Let $M$ be such that $K^n(M)$ is nonstandard, then $K^n(M)$ is not an initial segment of $M$, since a $\Sigma_n$-elementary initial segment of a model of $PA$ satisfies $B\Sigma_{n+1}$. Let $I$ be the initial segment generated by $K^n(M)$, i.e.

$$I := \{ a \in M : \exists b \in K^n(M) \ a < b \} .$$

Then if $n \geq 2$, $I$ is closed since $I \models I\Sigma_{n-1}$. In the case $n = 1$, replace $I$ by the smallest initial segment containing $K^n(M)$ that is closed under exponentiation.

Define a satisfaction class

$$S := \{ (\varphi, a) : \varphi \in I \text{ and } M \models Sat_{\Sigma_n}(\varphi, a) \} ,$$

which has the desired properties simply by definition.

Then obviously $K^n(M) \not\subseteq K^n_{\beta}(M, S)$, since $K^n(M) \not\subseteq I$. On the other hand, the above results imply that $I \not\supseteq K^n_{\beta}(M, S)$, since $K^n_{\beta}(M, S)$ cannot be an initial segment of $M$.

**Acknowledgement:** I like to thank Roman Murawski for invoking my interest in satisfaction classes, and for some discussion about the contents of this paper.

**References**

