On Threshold Logic and Cutting Planes Proofs

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Yet Another Formulation of Propositional Threshold Logic

Let $PTK^*$ be defined like $PTK$ in [1, 2], but with the rule $T^*_k$-right replaced by the two rules

$$T^*_k\text{-right1} : \frac{\Gamma \Rightarrow A_1, \Delta}{\Gamma \Rightarrow T_{k-1}^n(A_2, \ldots, A_n), \Delta}$$
$$T^*_k\text{-right2} : \frac{\Gamma \Rightarrow T_{k-1}^n(A_2, \ldots, A_n), \Delta}{\Gamma \Rightarrow T_k^n(A_1, \ldots, A_n), \Delta}$$

and $T^*_k$-left replaced by the two dual rules

$$T^*_k\text{-left1} : \frac{A_1, \Gamma \Rightarrow \Delta}{T_k^n(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$
$$T^*_k\text{-left2} : \frac{T_{k-1}^n(A_2, \ldots, A_n), \Gamma \Rightarrow \Delta}{T_k^n(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}.$$

The correctness of $PTK^*$ is obvious, and the completeness follows from Theorem 1 below and the completeness of $PTK$. In the following, we show that $PTK$ and $PTK^*$ are polynomially equivalent, and that the mutual simulations also respect the depth of proofs. This was claimed without proof in [3], where $PTK^*$ was first defined.

**Theorem 1.** If $P$ is a proof in $PTK$, then there is a proof $P'$ in $PTK^*$ of the same end-sequent. The size of $P'$ is linear in the size of $P$, and the formula depths of $P$ and $P'$ are the same.

**Proof.** Each application of the rule $T^*_k$-right is replaced by a subproof that is built as follows: From the second premise we get by weakening the sequent

$$\Gamma \Rightarrow T^n_{k-1}(A_2, \ldots, A_n), T^n_{k-1}(A_2, \ldots, A_n), \Delta,$$
and from this and the first premise we get by an application of $T^n_k$-right1

$$ \Gamma \implies T^n_k(A_1, \ldots, A_n), T^{n-1}_k(A_2, \ldots, A_n), \Delta . $$

From this sequent we obtain the conclusion by structural inferences and $T^n_k$-right2. Likewise, each application of $T^n_k$-left is replaced by a similar, dual subproof. The size and depth bounds are obvious. $\square$

**Theorem 2.** If $P$ is a proof in $PTK^*$, then there is a proof $P'$ in $PTK$ of the same end-sequent. The size of $P'$ is polynomial in the size of $P$, and the formula depths of $P$ and $P'$ are the same.

**Proof.** First, each application of the rule $T^n_k$-right1 can be simulated by $T^n_k$-right of $PTK$ preceded by a weakening, and likewise $T^n_k$-left1 can be simulated using weakening and $T^n_k$-left.

In [2] it was noted that the sequents

$$ (\ast) \quad T^n_k(A_1, \ldots, A_m) \implies T^{n-1}_k(A_1, \ldots, A_m) $$

have proofs in $PTK$ of size polynomial in $m$. Using these, we can replace each application of $T^n_k$-right2 by a subproof constructed as follows: From the premise of $T^n_k$-right2 and an instance of $(\ast)$ we obtain

$$ \Gamma \implies T^{n-1}_k(A_2, \ldots, A_n), \Delta , $$

by a cut, and again from the premise of $T^n_k$-right2 we obtain by weakening

$$ \Gamma \Rightarrow A_1, T^{n-1}_k(A_2, \ldots, A_n), \Delta . $$

From these two we obtain the conclusion by $T^n_k$-right. A dual proof using $(\ast)$ can serve to replace applications of $T^n_k$-left2. The size bound holds if we see the two uses of the premise of $T^n_k$-right2 as identical, i.e. if the proof is not tree-like $\square$

Theorems 1 and 2 together imply that $PTK^*$ enjoys cut-elimination, as the subproofs used in the proof of Theorem 1 are cut-free. They are also tree-like, hence Theorem 1 also holds for cut-free and tree-like proofs. The subproofs used in the proof of Theorem 2 are, as noted, not tree-like, and use cuts. Hence a question is:

Do cut-free and/or tree-like $PTK$-proofs polynomially simulate cut-free / tree-like $PTK^*$-proofs?

Another problem is to improve the size bounds in Theorem 2.
Embedding Unary Cutting Planes into $PTK^*$

A Unary Cutting Planes ($CP^*$) inequality can be written in the form

$$\sum_{i=1}^{n} x_i - \sum_{i=n+1}^{n+m} x_i \geq k,$$

where $n, m \in \mathbb{N}$, $k \in \mathbb{Z}$ and the variables $x_1, \ldots, x_{n+m}$ are not necessarily distinct. By a result in [2], a $CP^*$-proof can be assumed to use only the axioms $x \geq 0$, $-x \geq -1$, addition and division by 2.

For convenience, let $T^n_r(A_1, \ldots, A_n)$ for $n \geq 0$ stand for $\top$, and $T^0_r()$ with $k > 0$ stand for $\bot$. Let $E$ denote the inequality above, then its translation $\hat{E}$ in $PTK$ is defined as

$$T^{n+m}_r(\bar{x}_1, \ldots, \bar{x}_n, \bar{x}_{n+1}, \ldots, \bar{x}_{n+m}),$$

where $r := \max(k + m, 0)$.

**Theorem 3.** Let $P$ be a $CP^*$-proof of an inequality $E$ from the inequalities $E_1, \ldots, E_n$. Then there is a $PTK^*$-proof of the sequent

$$\hat{E}_1, \ldots, \hat{E}_n \Rightarrow \hat{E}$$

of threshold depth 1 and size $O(|P|^0(1))$.

This implies that threshold depth 1 $PTK^*$-proofs can p-simulate $CP^*$ in the following sense:

**Corollary 4.** If $A$ is a tautology in DNF such that $\neg A$, written as a set of $CP^*$-inequalities, has a $CP^*$-refutation of size $s$, then there is a $PTK^*$-proof of $A$ of threshold depth 1 and size $O(s^{O(1)} + |A|)$.

**Proof.** Let $A$ be $\bigvee_{i \leq n} \bigwedge_{j \in I_i} \ell_{ij}$, then by the theorem there is a proof in $PTK^*$ of

$$\bigvee_{j \in J_1} \bar{\ell}_{ij}, \ldots, \bigvee_{j \in J_s} \bar{\ell}_{kj} \Rightarrow \bot$$

of threshold depth 1 and size $O(s^{O(1)})$. From this, $A$ can be derived trivially in size $O(|A|)$. \[\square\]

By Theorem 2, the same holds for $PTK$ in place of $PTK^*$. To prove Theorem 3, we first derive a series of lemmas. The first lemma is simple and can be proved by the reader.

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Lemma 5. There is a proof in PTK* of the sequent
\[ T^n_k(A_1, \ldots, A_n) \Rightarrow T^n_{k-1}(A_1, \ldots, A_n) \]
of threshold depth 1 and size \( O(n) \)

Here, as well as in the following lemmas, when we say a proof has threshold depth 1 we mean that its threshold depth is at most \( 1 + \) the maximal threshold depth of the subformulae \( A_i \). In particular, its threshold depth is 1 if the \( A_i \) do not contain any threshold connectives.

Lemma 6. There is a proof in PTK* of the equivalence
\[ T^{n+2}_{k+1}(A, \neg A, B_1, \ldots, B_n) \leftrightarrow T^n_k(B_1, \ldots, B_n) \]
of threshold depth 1 and size \( O(n) \).

Proof. Let \( \bar{B} \) abbreviate \( B_1, \ldots, B_n \). From the axioms \( T^n_k(\bar{B}) \Rightarrow T^n_k(\bar{B}) \)
and \( A \Rightarrow A \), we get the sequent
\[ T^{n+2}_{k+1}(A, \neg A, \bar{B}) \Rightarrow A, T^n_k(\bar{B}) \]
by \( T^n_k \)-left2 and then \( T^n_k \)-left1. In the same way using the axiom \( \neg A \Rightarrow \neg A \)
we get
\[ T^{n+2}_{k+1}(A, \neg A, \bar{B}) \Rightarrow \neg A, T^n_k(\bar{B}) \]
using \( T^n_k \)-left1 first and then \( T^n_k \)-left2. From these the sequent in the lemma follows by a cut. \( \square \)

Lemma 7. There is a proof in PTK* of the following equivalence, the
generalized De Morgan law
\[ \neg T^n_k(A_1, \ldots, A_n) \leftrightarrow T^n_{n-k+1}(\neg A_1, \ldots, \neg A_n) \]
of threshold depth 1 and size \( O(n^3) \).

Proof. For the direction from left to right, we have to derive the sequent
\( S_{n,k} := \Rightarrow T^n_k(A_1, \ldots, A_n), T^n_{n-k+1}(\neg A_1, \ldots, \neg A_n) \). First, we derive \( S_{n,n} \):
From the sequents \( \Rightarrow A_i, \neg A_i \) for \( 1 \leq i \leq n \), this is obtained by \( \land \)-right followed by \( \lor \)-right. Dually we get \( S_{n,1} \).
Now for \( 1 < k < n \), we derive \( S_{n,k} \) from \( S_{n-1,k} \) and \( S_{n-1,k-1} \) as follows: From \( \Rightarrow T^{n-1}_{k-1}(A_2, \ldots, A_n), T^{n-1}_{n-k+1}(\neg A_2, \ldots, \neg A_n) \) and the axiom \( A_1 \Rightarrow A_1 \), we derive
\[ A_1 \Rightarrow T^n_k(A_1, \ldots, A_n), T^n_{n-k+1}(\neg A_1, \ldots, \neg A_n) \]
by $T^n_k$-right1 and then $T^n_{k+1}$-right2. Likewise, from the axiom $\neg A_1 \equiv \neg A_1$ and $\iff T^n_{k-1}(A_2, \ldots, A_n), T^n_{n-k}(\neg A_2, \ldots, \neg A_n)$ we derive

$$\neg A_1 \Rightarrow T^n_k(A_1, \ldots, A_n), T^n_{n-k+1}(\neg A_1, \ldots, \neg A_n).$$

From these, $S_{n,k}$ is obtained by a cut.

Now a proof for $S_{n,k}$ is obtained by arranging the sequents $S_{i,j,d}$ for $1 \leq i \leq k$ and $0 \leq j \leq n - k$ in a rectangular matrix, where each sequent is proved from those to the left and above it, and those in the first row and column are derived directly. Thus, we get a proof of the direction from left to right that has $O(n^2)$ many sequents and is hence of size $O(n^3)$.

The direction from right to left is proved dually.

Lemma 8. For each permutation $\pi \in S_n$, there is a proof in $PTK^*$ of the sequent

$$T^n_k(A_1, \ldots, A_n) \Rightarrow T^n_k(A_{\pi(1)}, \ldots, A_{\pi(n)})$$

of threshold depth 1 and size $O(n^4)$.

Proof. We start by proving that the sequents

$$(*) \quad T^n_k(A, B, \bar{C}) \Rightarrow T^n_k(B, A, \bar{C})$$

have proofs of threshold depth 1 and size $O(n)$. First, using the axioms $T^n_{k-2}^{n-2}(\bar{C}) \Rightarrow T^n_{k-2}^{n-2}(\bar{C})$ as well as $A \Rightarrow A$ and $B \Rightarrow B$ we derive

$$\bar{A}, B, T^n_k(A, B, \bar{C}) \Rightarrow T^n_k(B, A, \bar{C})$$

for each choice of $\bar{A} = A$ or $\neg A$ and $\bar{B} = B$ or $\neg B$, which is easily done. From these, $(*)$ is obtained by several cuts. This proof uses constantly many steps, hence is of size $O(n)$.

Next we prove the lemma for special permutations consisting of one cycle of the form $(p, p-1, \ldots, 1)$: the sequents

$$(**) \quad T^n_k(A_1, \ldots, A_n) \Rightarrow T^n_k(A_p, A_1, \ldots, A_{p-1}, A_{p+1}, \ldots, A_n)$$

have proofs of threshold depth 1 and size $O(n^3)$. Note that the sequent $(**)$ is easily derived for $k = n$ and $k = 1$ using structural inferences, and for $p = 2$ it is just an instance of the sequent $(*)$ above.

Next we derive $(**)$ from the two sequents

$$T^n_{j-1}(A_2, \ldots, A_n) \Rightarrow T^n_{j-1}(A_p, A_2, \ldots, A_{p-1}, A_{p+1}, \ldots, A_n)$$

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for \( j = k, k-1 \) and \( A_1 \implies A_1 \), using first the \( T_k^n \)-rules and a cut to add \( A_1 \) on both sides, and then an instance of (**) and a cut to swap \( A_1 \) and \( A_p \) in the succedent.

Using these, an inductive proof of (**) can be built as a rectangular matrix as in the proof of Lemma 7, and like there the size of the resulting proof will be \( O(n^3) \).

For the general case, note that any permutation \( \pi \in S_n \) can be factored into at most \( n \) cycles of the above type, hence we get a proof for a general \( \pi \) by at most \( n-1 \) cuts from instances of the special case above, which gives a proof of size \( O(n^4) \). \( \square \)

**Lemma 9.** The rule \( T_k^n \)-right2 of \( PTK' \)

\[
\frac{\Gamma \implies T_k^n(A_1, \ldots, A_n), \Delta}{\Gamma \implies T_k^n(B_1, \ldots, B_m), \Delta} \quad \frac{\Gamma \implies T_{k+\ell}^m(A_1, \ldots, A_n, B_1, \ldots, B_m), \Delta}{= \implies B_i}
\]

can be simulated in \( PTK^\ast \) by a proof of threshold depth 4 and size \( O(m^2(m+n)^4) \).

**Proof.** We give a proof of the sequent \( S_{m,\ell} \) defined as

\[
T_k^n(A_1, \ldots, A_n), T_{\ell}^m(B_1, \ldots, B_m) \implies T_{k+\ell}^{n+m}(A_1, \ldots, A_n, B_1, \ldots, B_m),
\]

then the claim follows by using cuts. First we derive the sequents \( S_{m,m} \) from the axioms \( T_k^n(A_1, \ldots, A_n) \implies T_k^n(A_1, \ldots, A_n) \) and \( B_i \implies B_i \) for \( 1 \leq i \leq m \) giving

\[
T_k^n(A_1, \ldots, A_n), T_{m}^m(B_1, \ldots, B_m) \implies T_{k+m}^{n+m}(B_1, \ldots, B_m, A_1, \ldots, A_n)
\]

from which we get \( S_{m,m} \) by Lemma 8. The size of this proof is dominated by the size of the proof from Lemma 8, hence it is of size \( O((m+n)^4) \).

Similarly from \( T_k^n(A_1, \ldots, A_n) \implies T_k^n(A_1, \ldots, A_n) \) and \( B_i \implies B_i \), we get

\[
T_k^n(A_1, \ldots, A_n), B_i \implies T_{k+1}^{n+m}(A_1, \ldots, A_n, B_1, \ldots, B_m)
\]

for each \( 1 \leq i \leq m \), hence a \( \lor \)-left yields \( S_{m,1} \). This proof consists of \( m \) subproofs, each using a proof obtained from Lemma 8, so it is of size \( O(m(m+n)^4) \).

Now we show how to derive \( S_{m,\ell} \) from \( S_{m-1,\ell-1} \) and \( S_{m-1,\ell} \), then a proof of \( S_{m,\ell} \) is built as in the proof of Lemma 7. First from \( S_{m-1,\ell} \) (with the variables \( B_1, \ldots, B_m \)) and \( B_1 \implies B_1 \) we obtain

\[
T_k^n(A_1, \ldots, A_n), T_{\ell}^m(B_1, \ldots, B_m) \implies B_1, T_{k+\ell}^{n+m}(A_1, \ldots, A_n, B_1, \ldots, B_m)
\]

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On the other hand, from \(S_{m-1,\ell-1}\) and \(B_1 \implies B_1\) we obtain
\[
T^n_k(A_1, \ldots, A_n), T^m_\ell(B_1, \ldots, B_m), B_1 \implies T^{n+m}_k(A_1, \ldots, A_n, B_1, \ldots, B_m)
\]
Hence we obtain \(S_{m,\ell}\) by a cut.

The whole proof of \(S_{m,\ell}\) consists of \(O(m^2)\) many proofs of size \(O((m+n)^4)\), plus \(O(m)\) proofs of sequents \(S_{i,i}\) and \(S_{i,1}\) whose size is negligible, hence its size is \(O(m^2(m+n)^4)\).

\[\square\]

**Proof of Theorem 3.** By induction on the number of inferences in \(P\). If this number is 1, then \(P\) consists only of the inequality \(E\), and either \(E = E_i\) for some \(1 \leq i \leq n\), or \(E\) is a \(CP^*\)-axiom \(x \geq 0\) or \(-x \geq -1\). In either of these cases, the claim is trivial. Otherwise, \(P\) has a last inference, and we have to distinguish whether this is an addition or a division inference.

Let the last inference be an addition whose premises are
\[
\sum_{i=1}^n x_i - \sum_{i=n+1}^{n+m} x_i \geq k \quad \text{and} \quad \sum_{i=1}^p y_i - \sum_{i=p+1}^{p+q} y_i \geq \ell
\]
and whose conclusion is
\[
\sum_{i=1}^s z_i - \sum_{i=n+1}^{s+t} z_i \geq k + \ell,
\]
with \(s = n+p-c\) and \(t = m+q-c\), where \(c\) is the number of cancellations in the inference. We treat only the case where \(k+m \geq 0\) and \(\ell+q \geq 0\). So from the translations of the premises we get by Lemma 9
\[
T^{n+m+p+q}_{k+\ell+m+q}(x_1, \ldots, x_n, \neg x_{n+1}, \ldots, \neg x_{n+m}, y_1, \ldots, y_p, \neg y_{p+1}, \ldots, \neg y_{p+q}).
\]
By Lemma 8 we can sort the arguments such that all possible cancellations can be made by \(c\) applications of Lemma 6. After that the arguments can be sorted using Lemma 8 such that the result is
\[
T^{s+t}_{k+\ell+t}(z_1, \ldots, z_s, \neg z_{s+1}, \ldots, \neg z_{s+t}),
\]
which is the translation of the conclusion of the addition inference.

For the case of division, suppose we have
\[
T^{2n}_k(A_1, A_1, A_2, A_2, \ldots, A_n, A_n).
\]
We want to derive $T^n_{\lfloor \frac{k}{2} \rfloor}(A_1, A_2, \ldots, A_n)$, so for sake of contradiction, assume $\neg T^n_{\lfloor \frac{k}{2} \rfloor}(A_1, A_2, \ldots, A_n)$. By Lemma 7, we get

$$T^n_{n-\lfloor \frac{k}{2} \rfloor+1}(\neg A_1, \neg A_2, \ldots, \neg A_n)$$

and adding this to itself using Lemmas 9 and 8, we obtain

$$T^{2n}_{2n-2\lfloor \frac{k}{2} \rfloor+2}(\neg A_1, \neg A_1, \neg A_2, \ldots, \neg A_n, \neg A_n).$$

Using Lemma 7 again yields

$$\neg T^{2n}_{2\lfloor \frac{k}{2} \rfloor-1}(A_1, A_1, A_2, \ldots, A_n, A_n),$$

and since $2\lfloor \frac{k}{2} \rfloor - 1 \leq k$, we get a contradiction by using Lemma 5. This argument can be formalized in $PTK^*$ using cuts.

By the size and depth bounds for the lemmas used, the whole proof is of threshold depth 1 and of size polynomial in the size of the proof $P$. \qed

References

