A Note on Sharply Bounded Arithmetic

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Abstract

We prove some independence results for the bounded arithmetic theory R^0_2, and we define a class of functions that is shown to be an upper bound for the class of functions definable by a certain restricted class of Σ^b_1-formulae in extensions of R^0_2.

Introduction

We deal with fragments of the theory S_2 of Bounded Arithmetic of Buss [1], and assume that the reader is familiar with this work. Just like among the fragments of Peano Arithmetic, the weak fragments below $I\Sigma_1$ are the most interesting ones, the bottom levels of the various hierarchies of subtheories of S_2 leave a lot of seemingly difficult open questions. So e.g. the question whether $\Sigma^b_0 - PIND$ and $\Sigma^b_0 - LIND$ are equivalent over the $BASIC$ axioms, or even whether S^0_2 is a subtheory of T^0_2, are — to the author's knowledge — not answered yet. We know, however, from [5] that if S^0_2 is included in T^0_2, then the inclusion is proper. In this paper we consider fragments slightly stronger than S^0_2, but weaker than T^0_2.

In [4], we defined the extension S^0_{2+} of S^0_2, which has the additional function symbols P (for the predecessor), $\dot{-}$, MSP and $Count$, where $MSP(a,i)$ is the number obtained by cutting off the last i bits of a, and $Count(a)$ is the number of bits set in the binary expansion of a. The axioms of S^0_{2+} are the $BASIC$ axioms of [1] together with the following axioms on the new function symbols

- $P0 = 0, \quad P(Sx) = x \ , \quad x > 0 \rightarrow S(Px) = x$
- $x \dot{-} 0 = x \ , \quad x \dot{-} Sy = P(x \dot{-} y) \ , \quad x \geq y \rightarrow (x \dot{-} y) + y = x \ , \quad x < y \rightarrow x \dot{-} y = 0$
- $MSP(x,0) = x \ , \quad MSP(x,Si) = \lfloor \frac{1}{2} MSP(x,i) \rfloor$
- $Count(0) = 0 \ , \quad Count(2x) = Count(x) \ , \quad Count(S(2x)) = S(Count(x))$

and $\Sigma_0^b - PIND$ (for sharply bounded formulae in the extended language). For S_{2+}^0, we have the following independence results:

Theorem 1 *The function $\lfloor \frac{1}{3}x \rfloor$ cannot be Σ_1^b-defined in S_{2+}^0. Furthermore, there are even functions in the complexity class AC^0 not Σ_1^b-definable in S_{2+}^0.*

Proof: We give a sketch of the proof, for details see [4]. We interpret S_{2+}^0 in S_2 as follows: The domain of the interpretation are the sequence numbers of sequences in which every term is positive. The empty sequence interprets 0, and if $\langle a_1, \dots, a_n \rangle$ interprets a, then $\langle a_1, \dots, a_n, a_{n+1} \rangle$ interprets $a \cdot 2^{a_{n+1}}$ if n is odd and $(a+1) \cdot 2^{a_{n+1}} - 1$ if n is even. Then the interpretations of the primitive functions of S_{2+}^0 are polynomial time computable and hence Σ_1^b-defined in S_2, and S_2 proves the interpretation of every theorem of S_{2+}^0.

Now the sequence $\langle n+1 \rangle$ interprets $2^{n+1} - 1$, and the interpretation of $\lfloor \frac{1}{3}(2^{n+1} - 1) \rfloor$ is $\langle 1, \dots, 1 \rangle$, a sequence of length n with a sequence number greater than 2^n. Thus the provability of the interpretation of $\forall x \, \exists y \; y = \lfloor \frac{1}{3}x \rfloor$ in S_2 would contradict Parikh's Theorem. The same holds if we consider the function $\lfloor \frac{1}{3}(2^{|x|} - 1) \rfloor$ instead, which is easily seen to be in AC^0. □

For many purposes, the $LIND$ axioms are more convenient than the $PIND$ axioms. Therefore let L_{2+}^0 be like S_{2+}^0, only with $\Sigma_0^b - PIND$ replaced by $\Sigma_0^b - LIND$. Then we have

Proposition 2 *S_{2+}^0 and L_{2+}^0 are equivalent.*

The proofs of the analogous statements (Thms. 2.6 and 2.12) in [1] can be carried out in exactly the same way in our case. To prove $LIND$ for a formula $A(x)$ in S_{2+}^0, use $PIND$ on the formula $A(|x|)$. Similarly, to prove $PIND$ for $B(x)$ in L_{2+}^0, use $LIND$ on x in the formula $B(MSP(a, |a| \dot{-} x))$.

The theory R_2^0

In [6], the theories R_2^i in the language of S_2 augmented by $\dot{-}$ and MSP were defined. R_2^i is axiomatized by the $BASIC$ axioms, the above axioms for $\dot{-}$ and MSP, the extensionality axiom

$$|a| = |b| \wedge \forall i < |a| \, (Bit(a,i) = Bit(b,i)) \rightarrow a = b \; ,$$

where Bit is defined by $Mod2(a) := a \dot{-} 2\lfloor \frac{1}{2}a \rfloor$ and $Bit(a,i) := Mod2(MSP(a,i))$, and the $\Sigma_i^b - LBIND$ axioms

$$A(0) \wedge \forall x \, (A(\lfloor \frac{1}{2}x \rfloor) \rightarrow A(x)) \rightarrow \forall x \; A(|x|)$$

for every Σ_i^b formula $A(x)$. R_2^1 corresponds to the complexity class NC, since in [6] it is shown that R_2^1 is equivalent to the theory TNC of [3], whose Σ_1^b-definable functions are exactly those in NC.

We shall mainly be interested in R_2^0, since our results about S_{2+}^0 can be applied to this theory. What is needed for this application is the following

Theorem 3 *The extensionality axiom can be proved in S_{2+}^0.*

Proof: Let $B(x)$ be the formula

$$\begin{aligned} |a| = |b| \wedge \forall i \leq |a| \, (i \leq x \rightarrow Bit(a, |a| \dot{-} i) = Bit(b, |a| \dot{-} i)) \\ \rightarrow \quad MSP(a, |a| \dot{-} x) = MSP(b, |a| \dot{-} x) \ . \end{aligned}$$

Then we can trivially prove $B(0)$ in R_2^0. Now suppose $B(x)$, and furthermore suppose

$$\forall i \leq |a| \, (i \leq Sx \rightarrow Bit(a, |a| \dot{-} i) = Bit(b, |a| \dot{-} i)) \ .$$

The latter formula is equivalent to the conjunction of $\forall i \leq |a| \, (i \leq x \rightarrow Bit(a, |a| \dot{-} i) = Bit(b, |a| \dot{-} i))$ and $Bit(a, |a| \dot{-} Sx) = Bit(b, |a| \dot{-} Sx)$, and by the hypothesis $B(x)$, we conclude $MSP(a, |a| \dot{-} x) = MSP(b, |a| \dot{-} x)$. The following equations are immediately proved from the definition of Bit without induction:

$$\begin{aligned} MSP(a, |a| \dot{-} Sx) &= 2 \cdot MSP(a, |a| \dot{-} x) + Bit(a, |a| \dot{-} Sx) \quad \text{and} \\ MSP(b, |a| \dot{-} Sx) &= 2 \cdot MSP(b, |a| \dot{-} x) + Bit(b, |a| \dot{-} Sx) \ . \end{aligned}$$

By the above, the terms on the right sides of these equations are equal, hence

$$MSP(a, |a| \dot{-} Sx) = MSP(b, |a| \dot{-} Sx) \ ,$$

which proves $B(Sx)$. Hence $R_2^0 \vdash B(x) \rightarrow B(Sx)$, and by $\Sigma_0^b - LIND$ we can conclude $B(|a|)$, which is equivalent to the extensionality axiom. □

Corollary 4 *The theory obtained from S_{2+}^0 by omitting the function symbol $Count$ and the axioms containing it is equivalent to R_2^0.*

Proof: In [6] it was shown that R_2^0 is equivalent to the theory obtained by adding to S_2^0 the functions $\dot{-}$ and MSP with their defining axioms and the extensionality axiom. Clearly the function P and the axioms containing it are redundant in S_{2+}^0, and since in the proof of Thm. 3 the function $Count$ is not used, the claim follows. □

By Thm. 1, we know that there are functions in the class AC^0 which are not Σ_1^b-definable in S_{2+}^0. Obviously, this also holds for the subsystem without the function $Count$, hence we have

Corollary 5 R_2^0 *cannot* Σ_1^b*-define every function in* AC^0

The following consequence of Thm. 1 was also observed by G. Takeuti (in a letter to the author).

Theorem 6 S_{2+}^0 *does not prove the* Σ_0^b*-comprehension axioms*

$$\exists y < 2^{|a|}\, \forall i < |a|\, (Bit(y,i) = 1 \leftrightarrow A(i))$$

for all sharply bounded formulae $A(i)$.

Proof: The theory T^0AC^0 defined in [2] is essentially the same as S_{2+}^0 together with the extensionality and Σ_0^b-comprehension axioms, but in a language without *Count* and multiplication, which is replaced by a restricted multiplication of the form $2^{|x|} \cdot y$. Hence if the Σ_0^b-comprehension axioms could be proved in S_{2+}^0, then T^0AC^0 would be a subtheory of S_{2+}^0.

But by Thm. 33 of [2], the Σ_1^b-definable functions of T^0AC^0 are exactly the functions in AC^0, hence every function in AC^0 would be Σ_1^b-definable in S_{2+}^0, contrary to Thm. 1. □

Corollary 7 R_2^0 *does not prove all* Σ_0^b*-comprehension axioms.*

Since the class of sharply bounded formulae is closed under negation, this corollary contrasts with the fact (cf. [6]) that for $i \geq 1$, R_2^i proves the Δ_i^b-comprehension axioms

$$\forall i\, (A(i) \leftrightarrow \neg B(i)) \rightarrow \exists y < 2^{|a|}\, \forall i < |a|\, (Bit(y,i) = 1 \leftrightarrow A(i))$$

for every pair of Σ_i^b-formulae $A(i)$ and $B(i)$.

The proof of Thm. 3 also shows that the extensionality axiom can be omitted from the theories TAC^0 and T^0AC^0 of [2] and their extensions.

$p\Sigma_1^b$-definable functions of S_{2+}^0 and R_2^0

Following Clote and Takeuti [2], we define the class of *pure* Σ_1^b*-formulae*, or $p\Sigma_1^b$-formulae for short, as follows:

Definition: A $p\Sigma_1^b$-formula is a formula of the form

$$\exists x_1 \leq t_1\ \ldots \exists x_n \leq t_n\ A(x_1, \ldots, x_n)$$

where $A(x_1, \ldots, x_n)$ is sharply bounded. The notion of a $p\Sigma_1^b$-definable function in a theory T is defined analogous to that of a function being Σ_1^b-definable in T.

Note that Σ^b_1-replacement implies that every Σ^b_1-formula is equivalent to a $p\Sigma^b_1$-formula. In particular, every predicate definable in the standard model by a Σ^b_1-formula can also be defined by a $p\Sigma^b_1$-formula. We expect that the class of $p\Sigma^b_1$-definable functions in S^0_{2+} and R^0_2 does not differ much from the class of Σ^b_1-definable functions, although we suspect that Σ^b_1-replacement cannot be proved in S^0_{2+}. Evidence for this is supported by the fact that S^0_{2+} does not prove the following weak form of Σ^b_1-replacement

$$\forall x < |a| \, \exists y \leq 1 \, B(x,y) \rightarrow \exists y < 2^{|a|} \, \forall i < |a| \, B(i, Bit(y,i))$$

for all sharply bounded $B(x,y)$, since it implies Σ^b_0-comprehension: to prove the comprehension axiom for a sharply bounded formula $A(x)$, let $B(x,y) :\leftrightarrow (y = 1 \leftrightarrow A(x))$ in the above schema[1].

Definition: Let $f_1, \ldots, f_k$ be some functions. The class $\mathcal{C}[f_1, \ldots, f_k]$ is the smallest class of functions containing

$$c^{(0)}_0, c^{(1)}_0, S, \pi^{(k)}_i, +, \cdot, \dot{-}, \lfloor \frac{1}{2} . \rfloor, |.|, \#, MSP \text{ and } f_1, \ldots, f_k$$

where $c^{(i)}_0$ is the i-ary constant zero, and $\pi^{(k)}_i(x_1, \ldots, x_k) = x_i$, and closed under composition and *sharply bounded minimization*, i.e. if g is in $\mathcal{C}[f_1, \ldots, f_k]$, then the function

$$\mu x < |a| \, (f(x, \underline{b}) = 0) \; := \; \begin{cases} \text{the least } x \text{ with } f(x, \underline{b}) = 0 & \text{if } \exists x < |a| \, f(x, \underline{b}) = 0 \\ |a| & \text{else} \end{cases}$$

is also in $\mathcal{C}[f_1, \ldots, f_k]$. If $k = 0$, the resulting class is simply called $\mathcal{C}$.

The class $\mathcal{C}[Count]$ is properly contained in the complexity class $NC^1 = ALogTIME$, and even in the probably smaller class TC^0. Furthermore, if in the definition of $\mathcal{C}$ multiplication would be removed from the set of initial functions, then the resulting class would be a proper subclass of AC^0. But even with multiplication and the function $Count$, we do not obtain all of AC^0, i.e. the difference $AC^0 \setminus \mathcal{C}[Count]$ is non-empty. This can be proved like Thm. 1 by the method of [4]. Therefore we consider the classes $\mathcal{C}[f_1, \ldots, f_k]$ as being very small.

We shall show that the $p\Sigma^b_1$-definable functions of R^0_2 are all in $\mathcal{C}$, and the $p\Sigma^b_1$-definable functions of S^0_{2+} are all in $\mathcal{C}[Count]$. Before we can do this, a little bootstrapping of the classes $\mathcal{C}[f_1, \ldots, f_k]$ is needed. As usual, we say that a predicate A is in $\mathcal{C}[f_1, \ldots, f_k]$ if its characteristic function χ_A is.

Proposition 8 *The ordering relation $\leq$ is in $\mathcal{C}[f_1, \ldots, f_k]$, and the class of predicates in $\mathcal{C}[f_1, \ldots, f_k]$ is closed under boolean operations and sharply bounded quantification. Finally, $\mathcal{C}[f_1, \ldots, f_k]$ is closed under definition by cases.*

[1] This consequence of Thm. 6 was pointed out by the referee.

Proof: Define $\overline{sg}(x) := 1 \dot{-} x$, then $\chi_{\leq}(x,y) := \overline{sg}(x \dot{-} y)$. Furthermore, $\overline{sg}$ yields the closure under negation, and closure under conjunction is simply obtained by multiplying the characteristic functions. For closure under quantification, simply note that

$$\forall x \leq |t|\ A(x) \quad \Leftrightarrow \quad \mu x < |t| + 1\, \neg A(x) \;=\; |t|\ .$$

Finally define the function $f(x) =$ if $A(x)$ then $g_1(x)$ else $g_2(x)$ by

$$f(x) := \chi_A(x) \cdot g_1(x) + \chi_{\neg A}(x) \cdot g_2(x)\ .$$

By Corollary 4 above, we can think of R^0_2 as the fragment of S^0_{2+} without $Count$, axiomatized in a sequent calculus like defined in [1, Ch. 4] with the $\Sigma^b_0 - LIND$ rule, and of S^0_{2+} as the extension $R^0_2[Count]$. In general, let $R^0_2[f_1,\dots,f_k]$ be R^0_2 extended by the function symbols $f_1,\dots,f_k$ with some quantifier-free axioms uniquely specifying them in the standard model, and $LIND$ for sharply bounded formulae in the extended language.

By a standard proof theoretic argument, we can assume that every formula in a proof of $\exists y \leq t\ A(a,y)$ with A a $p\Sigma^b_1$-formula is $p\Sigma^b_1$. Therefore our intended result follows from the following witnessing theorem for $p\Sigma^b_1$-formulae:

Theorem 9 *Let $C_i(\underline{a})$ be the $p\Sigma^b_1$-formula*

$$\exists x_{i1} \leq t_{i1}\ \dots \exists x_{ik_i} \leq t_{ik_i}\ A_i(\underline{x_i}, \underline{a})\ ,$$

where $\underline{x_i}$ denotes the sequence $x_{i1},\dots,x_{ik_i}$, and let $D_j(\underline{a})$ be the $p\Sigma^b_1$-formula

$$\exists y_{j1} \leq s_{j1}\ \dots \exists y_{j\ell_j} \leq s_{j\ell_j}\ B_j(\underline{y_j}, \underline{a})\ ,$$

and let $R^0_2[f_1,\dots,f_k]$ prove the following sequent

$$C_1(\underline{a}),\dots,C_n(\underline{a}) \implies D_1(\underline{a}),\dots,D_m(\underline{a})$$

where the formulae A_i, B_j are sharply bounded, and all the free variables in the sequent are among the $\underline{a}$. Then there are functions g_{ij}, $1 \leq i \leq m$, $1 \leq j \leq \ell_i$ in $\mathcal{C}[f_1,\dots,f_k]$ such that

$$\begin{aligned} & b_{11} \leq t_{11}, \dots, b_{1k_1} \leq t_{1k_1},\ A_1(\underline{b_1}, \underline{a}), \dots, b_{n1} \leq t_{n1}, \dots, b_{nk_n} \leq t_{nk_n},\ A_n(\underline{b_n}, \underline{a}) \\ & \quad \implies g_{11}(\underline{b}, \underline{a}) \leq s_{11} \wedge \dots \wedge g_{1\ell_1}(\underline{b}, \underline{a}) \leq s_{1\ell_1} \wedge B_1(g_{11}(\underline{b}, \underline{a}), \dots, g_{1\ell_1}(\underline{b}, \underline{a}), \underline{a}), \dots \\ & \qquad \dots, g_{m1}(\underline{b}, \underline{a}) \leq s_{m1} \wedge \dots \wedge g_{m\ell_m}(\underline{b}, \underline{a}) \leq s_{m\ell_m} \wedge B_m(g_{m1}(\underline{b}, \underline{a}), \dots, g_{m\ell_m}(\underline{b}, \underline{a}), \underline{a}) \end{aligned}$$

is satisfied in the standard model, where $\underline{b}$ denotes the sequence of all the variables b_{ij}.

Proof: This is an adaption of the proof of Thm. 24 in [2], by induction on the length of a proof of the sequent from the theorem, which we abbreviate $\Gamma \implies \Delta$.

If $\Gamma \implies \Delta$ is an initial sequent, then there is nothing to prove since we assumed that all the axioms are quantifier-free. Otherwise, we distinguish cases dependent on the last inference of a proof of $\Gamma \implies \Delta$. Most cases are straightforward, the only nontrivial ones being ($\exists \leq$:right), (Contraction:right), (Cut) and $\Sigma_0^b - LIND$. We shall in fact treat only simple cases of these inferences which show the principal ideas, which would be hidden behind technical details in a treatment of the general cases.

So let the last inference in the proof be ($\exists \leq$:right) of the form

$$\frac{\exists x \leq s_1\ A(\underline{a}, x) \implies \exists y \leq s_2\ B(\underline{a}, y, t(\underline{a}))}{t(\underline{a}) \leq u\ ,\ \exists x \leq s_1\ A(\underline{a}, x) \implies \exists z \leq u\ \exists y \leq s_2\ B(\underline{a}, y, z)}\ .$$

By the induction hypothesis we have a function g in $\mathcal{C}[f_1, \dots, f_k]$ such that

$$b \leq s_1\ ,\ A(\underline{a}, b) \implies g(\underline{a}, b) \leq s_2 \wedge B(\underline{a}, g(\underline{a}, b), t(\underline{a}))$$

is true. Then we can simply define the function $h(\underline{a}, b) := t(\underline{a})$, since every term in the language of $R_2^0[f_1, \dots, f_k]$ is in $\mathcal{C}[f_1, \dots, f_k]$, and obtain

$$t(\underline{a}) \leq u\ ,\ b \leq s_1\ ,\ A(\underline{a}, b) \implies h(\underline{a}, b) \leq u \wedge g(\underline{a}, b) \leq s_2 \wedge B(\underline{a}, g(\underline{a}, b), h(\underline{a}, b))\ .$$

Now let the last inference be a (Contraction:right), which we assume for sake of simplicity to look like

$$\frac{\exists x \leq s\ A(\underline{a}, x) \implies \exists y \leq t\ B(\underline{a}, y)\ ,\ \exists y \leq t\ B(\underline{a}, y)}{\exists x \leq s\ A(\underline{a}, x) \implies \exists y \leq t\ B(\underline{a}, y)}\ .$$

By the induction hypothesis, there are functions g_1 and g_2 in $\mathcal{C}[f_1, \dots, f_k]$ such that

$$b \leq s, A(\underline{a}, b) \implies g_1(\underline{a}, b) \leq t \wedge B(\underline{a}, g_1(\underline{a}, b))\ ,\ g_2(\underline{a}, b) \leq t \wedge B(\underline{a}, g_2(\underline{a}, b))$$

is true. Define the function g by

$$g(\underline{a}, b) := \begin{cases} g_1(\underline{a}, b) & \text{if } g_1(\underline{a}, b) \leq t \wedge B(\underline{a}, g_1(\underline{a}, t)) \\ g_2(\underline{a}, b) & \text{else} \end{cases}\ .$$

By Prop. 8, g is in $\mathcal{C}[f_1, \dots, f_k]$, and obviously we have

$$b \leq s, A(\underline{a}, b) \implies g(\underline{a}, b) \leq t \wedge B(\underline{a}, g(\underline{a}, t))\ .$$

Now let the last inference be a (Cut), which we assume to look like

$$\frac{\exists x \leq t\ A(\underline{a}, x) \implies \exists y \leq s\ B(\underline{a}, y) \qquad \exists y \leq s\ B(\underline{a}, y) \implies \exists z \leq u\ C(\underline{a}, z)}{\exists x \leq t\ A(\underline{a}, x) \implies \exists z \leq u\ C(\underline{a}, z)}$$

By the induction hypothesis, there are functions g_1 and g_2 in $\mathcal{C}[f_1, \dots, f_k]$ such that

$$b \le t, A(\underline{a}, b) \implies g_1(\underline{a}, b) \le s \wedge B(\underline{a}, g_1(\underline{a}, b)) \qquad \text{and}$$

$$c \le s, B(\underline{a}, c) \implies g_2(\underline{a}, c) \le u \wedge C(\underline{a}, g_2(\underline{a}, c))$$

are true. Therefore we have

$$b \le t, A(\underline{a}, b) \implies g_2(\underline{a}, g_1(\underline{a}, b)) \le u \wedge C(\underline{a}, g_2(\underline{a}, g_1(\underline{a}, b))) \,.$$

Finally, let the last inference be a $\Sigma_0^b - LIND$ of the form

$$\frac{\exists x \le s\, B(\underline{a}, x)\,,\, A(\underline{a}, b) \implies A(\underline{a}, Sb)\,,\, \exists y \le t\, C(\underline{a}, y)}{\exists x \le s\, B(\underline{a}, x)\,,\, A(\underline{a}, 0) \implies A(\underline{a}, |c|)\,,\, \exists y \le t\, C(\underline{a}, y)}\,,$$

then by the induction hypothesis we have a function g in $\mathcal{C}[f_1, \dots, f_k]$ such that

$$d \le s\,,\, B(\underline{a}, d)\,,\, A(\underline{a}, b) \implies A(\underline{a}, Sb)\,,\, g(\underline{a}, d, b) \le t \wedge C(\underline{a}, g(\underline{a}, d, b))$$

is true. What we need is a function h such that

$$d \le s\,,\, B(\underline{a}, d)\,,\, A(\underline{a}, 0) \implies A(\underline{a}, |c|)\,,\, h(\underline{a}, d, c) \le t \wedge C(\underline{a}, h(\underline{a}, d, c))$$

is true. Define the function $h(\underline{a}, d, c) := g(\underline{a}, d, \mu x < |c|\, g(\underline{a}, d, x) \le t \wedge C(\underline{a}, g(\underline{a}, d, x)))$. Then there are two cases:

- There is an $x < |c|$ with $g(\underline{a}, d, x) \le t \wedge C(\underline{a}, g(\underline{a}, d, x))$. In this case, $h(\underline{a}, d, c) \le t \wedge C(\underline{a}, h(\underline{a}, d, c))$ is true.
- For all $x < |c|$, $g(\underline{a}, d, x) \le t \wedge C(\underline{a}, g(\underline{a}, d, x))$ is false, hence by the induction hypothesis we can conclude $A(\underline{a}, |c|)$ inductively from $A(\underline{a}, 0)$.

In either case, the sequent above is true. □

Corollary 10 *Every function $p\Sigma_1^b$-definable in $R_2^0[f_1, \dots, f_k]$ is in $\mathcal{C}[f_1, \dots, f_k]$.*

This follows immediately from Thm. 9.

Note that the only restriction imposed on the theories $R_2^0[f_1, \dots, f_n]$ is that the functions $f_1, \dots, f_n$ are axiomatized by quantifier-free axioms. Thus Thm. 9 and its corollary apply e.g. to the theories R_k^0 for $k > 2$, where $R_k^0 := R_2^0[\#_3, \dots, \#_k]$ and the functions $\#_i$ are defined by $\#_2 := \#$ and $x \#_{i+1} y := 2^{|x| \#_i |y|}$.

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