Automated Theorem Proving

Prof. Dr. Jasmin Blanchette, Yiming Xu, PhD,
Lydia Kondylidou, and Tanguy Bozec
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Exercises 10: Termination

Exercise 10.1: Let $\Sigma = (\{f/1,\ g/2,\ h/1,\ b/0,\ c/0\}, \emptyset)$ and let

$$
\begin{aligned}
t_1 &= g(h(x), h(c)),\\
t_2 &= g(x, x),\\
t_3 &= g(b, f(x)),\\
t_4 &= f(g(x, y)),\\
t_5 &= h(g(x, c)).
\end{aligned}
$$

Determine for each $1 \leq i < j \leq 5$ whether t_i and t_j are uncomparable or comparable (and if so, which term is larger) with respect to

(a) the lexicographic path ordering with the precedence $f \succ g \succ h \succ b \succ c$,

(b) the Knuth–Bendix ordering with the precedence $h \succ f \succ g \succ b \succ c$, where h has weight 0, b has weight 3, and all other symbols and variables have weight 1,

(c) the polynomial ordering over $\{n \in \mathbb{N} \mid n \geq 1\}$ with $P_f(X_1) = X_1 + 1$, $P_g(X_1, X_2) = 2X_1 + X_2 + 1$, $P_h(X_1) = 3X_1$, $P_b = 1$ and $P_c = 3$.

Exercise 10.2: Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering on Ω, and let $s, t \in \mathrm{T}_\Sigma(X)$.

(a) Prove: If s contains a subterm $s' = f(s_1, \dots, s_n)$ such that $\mathrm{var}(s') \supseteq \mathrm{var}(t)$ and $f \succ g$ for all function symbols g occurring in t, then $s \succ_{\mathrm{lpo}} t$.

(b) Refute: If s contains a subterm $s' = f(s_1, \dots, s_n)$ such that $\mathrm{var}(s) \supseteq \mathrm{var}(t)$ and $f \succ g$ for all function symbols g occurring in t, then $s \succ_{\mathrm{lpo}} t$.

Exercise 10.3: Determine for each of the following statements whether they are true or false:

(1) $f(g(x)) \succ f(x)$ in every simplification ordering $\succ$.

(2) $f(f(x)) \succ f(y)$ in every simplification ordering $\succ$.

(3) If $\succ$ is an lpo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.

(4) If $\succ$ is a kbo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.

(5) If $\succ$ is an lpo, then $h(f(x), y, y) \succ h(x, z, z)$.

(6) If $\succ$ is a kbo, then $h(f(x), f(y), z) \succ h(x, f(z), y)$.

(7) There is a reduction ordering $\succ$ such that $f(x) \succ g(f(x))$.

(8) There is a reduction ordering $\succ$ such that $f(f(x)) \succ f(g(f(x)))$.

Exercise 10.4 (*): Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let h be a unary function symbol in Ω, and let $\succ$ be precedence on Ω such that h is the smallest element of Ω w.r.t. $\succ$.

Prove: For all terms $s, t \in \mathrm{T}_\Sigma(X)$, we have $s \succ_{\mathrm{lpo}} t$ if and only if $s \succeq_{\mathrm{lpo}} h(t)$.

Exercise 10.5: Let $\Sigma = (\{f/2,\ g/2,\ h/2\}, \emptyset)$. Let R be the term rewrite system

$$\{\ g(x, f(x, y)) \to h(y, g(x, y)), \quad h(x, y) \to g(y, y)\ \}$$

Is there a lexicographic path ordering $\succ_{\mathrm{lpo}}$ such that $\to_R\ \subseteq\ \succ_{\mathrm{lpo}}$? If yes, give the precedence of this lpo; if no, explain why such an lpo does not exist.

Exercise 10.6: Let $\Sigma = (\{f/2,\ g/1,\ h/1, b/0\}, \emptyset)$. Let R be the term rewrite system

$$\{\ f(g(x), y) \to g(f(x, x)), \quad h(f(x, b)) \to g(x)\ \}$$

Is there a Knuth–Bendix ordering $\succ_{\mathrm{kbo}}$ such that $\to_R\ \subseteq\ \succ_{\mathrm{kbo}}$? If yes, give the weights and precedence of this kbo; if no, explain why such a kbo does not exist.

Exercise 10.7: Let $\Sigma = (\{f/1,\ g/1,\ b/0,\ c/0\}, \emptyset)$. Let R be the term rewrite system

$$\{\ f(g(x)) \to g(g(f(x))), \quad c \to f(b)\ \}$$

Is there a polynomial ordering $\succ_{\mathcal{A}}$ in which the function symbols are interpreted by linear polynomials over $U_{\mathcal{A}} = \{n \in \mathbb{N} \mid n \geq 1\}$ such that $\to_R\ \subseteq\ \succ_{\mathcal{A}}$? If yes, give the polynomials by which the symbols of Σ are interpreted; if no, explain why such an ordering does not exist.

Exercise 10.8: Let $\Sigma = (\Omega, \emptyset)$ be a finite signature. For $t \in \mathrm{T}_\Sigma(X)$ we define $\mathrm{depth}(t) = \max\{|p| \mid p \in \mathrm{pos}(t)\}$. Let $\succ$ be a strict partial ordering on Ω. The binary relation $\succ_{\mathrm{do}}$ on $\mathrm{T}_\Sigma(X)$ is defined by: $s \succ_{\mathrm{do}} t$ if and only if

(1) $\#(x,s) \geq \#(x,t)$ for all variables x and $\mathrm{depth}(s) > \mathrm{depth}(t)$, or

(2) $\#(x,s) \geq \#(x,t)$ for all variables x, $\mathrm{depth}(s) = \mathrm{depth}(t)$, and

   (a) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or

   (b) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)\ (\succ_{\mathrm{do}})_{\mathrm{lex}}\ (t_1, \ldots, t_m)$.

Give an example that shows that $>_{\mathrm{do}}$ is *not* a reduction ordering.

Exercise 10.9 (*)**:** Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let $\succ$ be a simplification ordering. Let R be a TRS over $\mathrm{T}_\Sigma(X)$ such that $l \succ r$ for all $l \to r \in R$. Let h be an n-ary function symbol in Ω (with $n > 0$) that does not occur in any left-hand side of a rule in R. Prove: If R is confluent, then $R \cup \{h(x, \ldots, x) \to x\}$ is confluent.

Exercise 10.10: Let $\Sigma = (\{f/1,\ g/2,\ h/2,\ b/0,\ c/0\}, \emptyset)$. Let E be the following set of equations over Σ:

$$f(f(x)) \approx g(b,x) \qquad (1)$$
$$h(f(y),y') \approx f(h(y,y')) \qquad (2)$$
$$g(h(z,z),c) \approx h(z,b) \qquad (3)$$

(a) Suppose that the three equations in E are turned into rewrite rules by orienting them from left to right. Give all critical pairs between the resulting three rules.

(b) It is possible to orient the equations in E using an appropriate kbo so that there are no critical pairs between the resulting rules. Give the weights and precedence of the kbo, and explain how the equations are oriented.

Exercise 10.11: Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$. Let N be the following set of clauses over Σ:

$$P(f(x), x) \lor P(c, x) \lor R(g(x), x) \qquad (1)$$
$$\neg P(y, f(y)) \qquad (2)$$
$$\neg P(y, c) \lor \neg P(z, h(y)) \lor Q(z) \qquad (3)$$
$$Q(b) \lor Q(x) \lor \neg R(g(x), x) \qquad (4)$$
$$R(g(c), y) \qquad (5)$$

(a) Suppose that the atom ordering $\succ$ is a lexicographic path ordering with the precedence $P \succ Q \succ R \succ f \succ g \succ h \succ b \succ c$ and that the selection function *sel* selects no literals. Compute all $\text{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the conclusions of the inferences computed in part (a) is redundant w.r.t. N. Which one? Why?

Exercise 10.12: Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$. Let N be the following set of clauses over Σ:

$$P(x, f(x)) \lor P(x, x) \qquad (1)$$
$$\neg P(h(z), x) \lor \neg P(y, f(f(x))) \lor \neg Q(x) \lor Q(f(x)) \qquad (2)$$
$$\neg Q(h(f(x))) \lor R(h(b), y) \qquad (3)$$
$$\neg R(y, g(c)) \lor Q(g(x)) \qquad (4)$$
$$\neg Q(h(y)) \qquad (5)$$

(a) Suppose that the atom ordering $\succ$ is a Knuth–Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ Q \succ R \succ f \succ g \succ h \succ b \succ c$ and that the selection function *sel* selects no literals. Compute all $\text{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the clauses (1)–(5) is redundant with respect to the others. Which one? Why? Give a brief explanation.

Exercise 10.13: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{f/1,\ b/0,\ c/0\}$ and $\Pi = \{P/1\}$. Suppose that the atom ordering $\succ$ is a Knuth–Bendix ordering with weight 1 for all predicate symbols, function symbols, and variables, and with the precedence $P \succ f \succ b \succ c$. Let $N = \{C_1, C_2, C_3\}$ with

$$\begin{aligned} C_1 &= P(b) \\ C_2 &= \neg P(f(f(c))) \\ C_3 &= P(x) \vee P(f(x)) \end{aligned}$$

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered with respect to the clause ordering $\succ_C$?

(b) Construct the candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ of the set of all ground instances of clauses in N. Which clauses in $G_\Sigma(N)$ are productive and what do they produce?

Exercise 10.14 (∗)**:** Let $\Sigma = (\{f/1,\, b/0,\, c/0\},\, \{P/1\})$. Let N be the following set of Σ-clauses:

$$P(b) \qquad (1)$$

$$P(f(c)) \qquad (2)$$

$$\neg P(x) \vee P(f(x)) \qquad (3)$$

Let $\succ$ be a Knuth–Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ f \succ b \succ c$. The ordering is extended to ground literals and ground clauses as usual. Give the smallest nonempty ground Σ-clauses C_1, C_2, C_3, C_4 such that

(a) $C_1 \in G_\Sigma(N)$ and $C_1 \in \mathrm{Red}(N)$,

(b) $C_2 \in G_\Sigma(N)$ and $C_2 \notin \mathrm{Red}(N)$,

(c) $C_3 \notin G_\Sigma(N)$ and $C_3 \in \mathrm{Red}(N)$,

(d) $C_4 \notin G_\Sigma(N)$ and $C_4 \notin \mathrm{Red}(N)$.