Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 7: Resolution Continued

Exercise 7.1: Find a strict total ordering $\succ$ on the ground atoms $P(b), P(c), Q, R$ such that

$$P(b) \lor \neg P(c) \succ_{\mathrm{C}} \neg P(b) \lor P(c) \qquad (1)$$

$$P(b) \lor P(b) \lor P(b) \lor R \succ_{\mathrm{C}} P(b) \lor R \lor R \qquad (2)$$

$$\neg P(b) \lor Q \succ_{\mathrm{C}} P(c) \lor R \qquad (3)$$

Exercise 7.2: Consider the following formulas:

$$F_1 = \forall x\, (S(x) \rightarrow \exists y\, (R(x,y) \land P(y)))$$

$$F_2 = \forall x\, (P(x) \rightarrow Q(x))$$

$$F_3 = \exists x\, S(x)$$

$$G = \exists x\, \exists y\, (R(x,y) \land Q(y))$$

Use ordered resolution to prove that $\{F_1, F_2, F_3\} \models G$. You may choose the selection function and the ordering on atoms.

Hint: You will need some preprocessing.

Exercise 7.3: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/2, Q/1, R/2\}$. Suppose that the atom ordering $\succ$ compares ground atoms by comparing lexicographically first the predicate symbols ($P \succ Q \succ R$), then the size of the first argument, then the size of the second argument (if present). If at least one of the two atoms to be compared is nonground, $\succ$ compares only the predicate symbols.

Let N be the following set of clauses:

$$P(f(x), x) \vee R(b, b) \qquad (1)$$

$$\neg P(b, x) \vee \neg P(x, b) \vee Q(x) \qquad (2)$$

$$Q(f(b)) \vee \neg Q(b) \vee R(f(x), b) \qquad (3)$$

$$Q(b) \vee \neg R(f(x), f(x)) \qquad (4)$$

$$\neg Q(x) \vee R(x, x) \qquad (5)$$

(a) Which literals are strictly maximal in the clauses of N?

(b) Which literals are maximal in the clauses of N?

(c) Which $Res^{\succ}_{sel}$-inferences are possible if *sel* selects no literals? What are their conclusions?

(d) Is there a $Res^{\succ}_{sel}$-inference between the clause

$$P(x, f(x)) \vee R(b, b) \qquad (1')$$

and clause (2) if *sel* selects no literals? Justify your answer.

(e) Define a selection function *sel* such that N is saturated under $Res^{\succ}_{sel}$.

Exercise 7.4: In Sect. 3.12 of the lecture notes, the inference rules for ground resolution with ordering restrictions (without selection functions) are given by

(Ground) Ordered Resolution:

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C} \quad \text{if } A \succ L \text{ for all } L \text{ in } D \text{ and } \neg A \succeq L \text{ for all } L \text{ in } C.$$

(Ground) Ordered Factorization:

$$\frac{C \vee A \vee A}{C \vee A} \quad \text{if } A \succeq L \text{ for all } L \text{ in } C.$$

This calculus is sound and refutationally complete for sets of ground clauses.

Suppose that we replace the ordering restriction for the first inference rule by "if $A \succ L$ for all L in D and $A \succeq L$ for all L in C."

(a) Is the calculus with this modification still sound? If yes, give a short explanation; if no, give a counterexample.

(b) Is the calculus with this modification still refutationally complete? If yes, give a short explanation; if no, give a counterexample.

Exercise 7.5: Determine all strict total orderings $\succ$ on the atomic formulas P, Q, R, S such that the associated clause ordering $\succ_{\mathrm{C}}$ satisfies the properties (1)–(3) simultaneously:

$$P \vee Q \succ_{\mathrm{C}} \neg Q \qquad (1)$$

$$R \vee Q \succ_{\mathrm{C}} \neg P \vee \neg P \qquad (2)$$

$$\neg R \vee \neg R \succ_{\mathrm{C}} S \qquad (3)$$

Exercise 7.6: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1,\ Q/1\}$. Suppose that the atom ordering $\succ$ compares ground atoms by comparing lexicographically first the size of the argument and then the predicate symbols ($P \succ Q$). Let N be the following set of clauses:

$$\neg P(x) \ \vee \ P(f(x)) \qquad (1)$$

$$\neg Q(f(b)) \ \vee \ P(f^3(b)) \qquad (2)$$

$$Q(b) \ \vee \ Q(f(b)) \qquad (3)$$

where $f^0(b)$ stands for b and $f^{i+1}(b)$ stands for $f(f^i(b))$.

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered w.r.t. the clause ordering $\succ_C$?

(b) Construct the candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ of the set of all ground instances of clauses in N. Is it a model of $G_\Sigma(N)$?

Exercise 7.7: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{b/0, f/1\}$ and $\Pi = \{P/1,\ Q/1\}$. Suppose that the atom ordering $\succ$ compares ground atoms by comparing lexicographically first the predicate symbols ($P \succ Q$) and then the size of the argument. Let N be the following set of clauses:

$$\neg Q(y) \vee P(y) \qquad (1)$$

$$Q(x) \vee Q(f(x)) \qquad (2)$$

(a) Sketch what the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered w.r.t. the clause ordering $\succ_C$?

(b) Construct the candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ of the set of all ground instances of clauses in N.

Exercise 7.8: Let N be a set of ground clauses, and let $\succ$ be a total and well-founded atom ordering. Prove or refute: If every clause in N is redundant w.r.t. N, then every clause in N is a tautology.

Exercise 7.9 (∗)**:** Give an example of two different first-order clauses F and G such that F entails G and G is not redundant w.r.t. $\{F\}$. If necessary, specify the atom ordering used by the redundancy criterion.

Exercise 7.10 (∗)**:** Give a clause C such that an "Ordered Resolution with Selection" inference is possible from C and C and the inference is not redundant w.r.t. $\{C\}$.