Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 5: Resolution

Exercise 5.1: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{P/1\}$. Determine for each of the following statements whether they are true or false:

(1) There is a Σ-model $\mathcal{A}$ of $P(b) \wedge \neg P(f(b))$ such that $U_{\mathcal{A}} = \{7, 8, 9\}$.

(2) There is a Σ-model $\mathcal{A}$ of $P(b) \wedge \neg P(f(f(b)))$ such that $f_{\mathcal{A}}(a) = a$ for every $a \in U_{\mathcal{A}}$.

(3) $P(b) \wedge \neg P(f(b))$ has a Herbrand model.

(4) $P(b) \wedge \forall x\, \neg P(x)$ has a Herbrand model.

(5) $\forall x\, P(f(x))$ has a Herbrand model with a two-element universe.

(6) $\forall x\, P(x)$ has exactly one Herbrand model.

(7) $\forall x\, P(f(x))$ entails $\forall x\, P(f(f(x)))$.

Exercise 5.2: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ-formula

$$\neg P(b) \ \wedge\ P(f(f(b))) \ \wedge\ \forall x\, (\neg P(x) \vee P(f(x))).$$

Determine for each of the following statements whether they are true or false:

(1) There is a Σ-model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.

(2) There is a Σ-model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = a$ for every $a \in U_{\mathcal{A}}$.

(3) F has exactly two Σ-models.

(4) Every Σ-model of F is a model of $\exists x\, P(x)$.

(5) Every Σ-model of F is a model of $\forall x\, P(f(f(x)))$.

(6) There are infinitely many Herbrand interpretations over Σ.

(7) There is a Herbrand model of F over Σ whose universe has exactly two elements.

(8) There is a Herbrand model of F over Σ with an infinite universe.

(9) F has exactly two Herbrand models over Σ.

Exercise 5.3: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{f/1,\ b/0,\ c/0\}$ and $\Pi = \{P/1\}$. Are the following statements correct?

(1) The formula $\forall x\, P(x)$ has infinitely many Σ-models.

(2) Every model of $\forall x\, P(x)$ is a model of $\forall x\, P(f(x))$.

(3) The formula $\neg P(b) \wedge \forall x\, P(x)$ has a Σ-model with an infinite universe.

(4) The formula $\neg P(b) \wedge \forall x\, P(f(x))$ has a Σ-model with a two-element universe.

(5) Every Σ-model of $P(b) \wedge P(c) \wedge \forall x\, P(f(x))$ is a model of $\forall x\, P(x)$.

(6) Every Herbrand model over Σ of $P(b) \wedge P(c) \wedge \forall x\, P(f(x))$ has an infinite universe.

(7) The formula $P(b) \vee P(c)$ has exactly three Herbrand models over Σ.

(8) The formula $\forall x\, P(f(x))$ has exactly four Herbrand models over Σ.

Exercise 5.4: Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ-formula

$$\neg P(b) \;\wedge\; P(f(f(b))) \;\wedge\; \forall x\, (P(x) \vee P(f(x))).$$

Determine for each of the following statements whether they are true or false:

(1) If $\mathcal{A}$ is a Σ-model of F, then $P_{\mathcal{A}} \neq \emptyset$ and $P_{\mathcal{A}} \neq U_{\mathcal{A}}$.

(2) There is a Σ-model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.

(3) There is a Σ-model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = f_{\mathcal{A}}(a')$ for all $a, a' \in U_{\mathcal{A}}$.

(4) F has exactly four Σ-models.

(5) There are infinitely many Herbrand interpretations over Σ.

(6) There is an Herbrand model of F over Σ with a finite universe.

(7) There is an Herbrand model $\mathcal{A}$ of F over Σ and an assignment β such that $\mathcal{A}(\beta)(f(b)) = \mathcal{A}(\beta)(f(f(b)))$.

Exercise 5.5: Determine for each of the following statements whether it is true or false:

(1) If $\Sigma = (\{b/0,\, c/0\},\, \{P/1\})$, then $P(b) \vee \neg P(c)$ has exactly three Herbrand models over Σ.

(2) If $\Sigma = (\{f/1,\, c/0\},\, \{P/1\})$, then $P(c) \vee P(f(c))$ has an Herbrand model over Σ whose universe has exactly four elements.

(3) If $\Sigma = (\{f/1,\, c/0\},\, \{P/1\})$, then $\neg P(c) \wedge \forall x\, P(f(x))$ has a model whose universe has exactly five elements

(4) If $\Sigma = (\{b/0,\, c/0,\, d/0\},\, \{P/1\})$, then $P(b) \vee \neg P(b)$ and $P(c) \vee \neg P(d)$ are equisatisfiable.

(5) If $\Sigma = (\{f/1,\, c/0\},\, \{P/1\})$, N is a set of universally quantified Σ-clauses, and every clause in N has at least one positive literal, then N has an Herbrand model.

(6) If $\Sigma = (\{f/1,\, c/0\},\, \{P/1\})$, N is a set of universally quantified Σ-clauses, and $N \models \neg P(x) \vee P(f(x))$, then N has a model.

(7) If $\Sigma = (\{f/1,\, c/0\},\, \{P/1\})$, then $\forall x\, P(f(x)) \models \forall y\, P(c) \vee P(f(f(y)))$.

Exercise 5.6: Let N be the set consisting of the following ground clauses:

$$
\begin{aligned}
&P \vee Q && (1)\\
&P \vee \neg Q && (2)\\
&\neg P \vee Q && (3)\\
&\neg P \vee \neg Q && (4)
\end{aligned}
$$

(a) Show that $N \vdash_{Res} \bot$, that is, derive $\bot$ from N using the "Resolution" and "Factorization" rules.

(b) Why is it impossible to derive the empty clause from N without using "Factorization"?

Exercise 5.7 (∗): Find a finite set N of ground clauses such that no clause in N is a tautology and such that $Res^*(N)$ is infinite.

Exercise 5.8: Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0,\ c/0\}$ and $\Pi = \{P/1,\ Q/0,\ R/0\}$. Use the ground resolution calculus *Res* to check whether the following clause set is satisfiable:

$$\begin{aligned} \neg P(b) \vee Q &\quad (1)\\ \neg P(b) \vee R &\quad (2)\\ \neg P(c) \vee Q &\quad (3)\\ \neg Q \vee \neg R &\quad (4)\\ Q \vee R &\quad (5)\\ P(b) &\quad (6)\\ \neg P(c) &\quad (7) \end{aligned}$$

Exercise 5.9: Use the ground resolution calculus to show that

$$\big\{(P \leftrightarrow (Q \wedge R)),\ (P \leftrightarrow Q)\big\} \models Q \rightarrow R$$

Hint: You will need some preprocessing.

Exercise 5.10: Prove or refute: $Res(N)$ is satisfiable if and only if N is satisfiable.

Exercise 5.11: Prove or refute: All clauses in $Res^*(N)$ are tautologies if and only if all clauses in N are tautologies.

Exercise 5.12 $(*)$**:** Prove the following statement: If N is a set of propositional formulas and C is a propositional formula such that $N \models C$, then there exists a finite subset $M \subseteq N$ such that $M \models C$.