Automated Theorem Proving

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Exercises 12: Superposition

Exercise 12.1: Refute the following set of equational clauses by superposition:

$$x \approx b \lor x \approx c \lor x \approx d \quad (1)$$
$$e \not\approx b \quad (2)$$
$$e \not\approx c \quad (3)$$
$$e \not\approx d \quad (4)$$

Choose an appropriate ordering and perform only inferences that satisfy the ordering restrictions.

Proposed solution. We use the lpo with the precedence $e \succ d \succ c \succ b$. First, we perform a "Negative Superposition" inference from (1) and (4) with the conclusion

$$e \approx b \lor e \approx c \lor d \not\approx d \quad (5)$$

Next, we perform an "Equality Resolution" inference from (5), yielding

$$e \approx b \lor e \approx c \quad (6)$$

Next, we perform "Negative Superposition" from (5) and (3), yielding

$$e \approx b \lor c \not\approx c \quad (7)$$

Next, we perform an "Equality Resolution" inference from (7), yielding

$$e \approx b \quad (8)$$

Next, we perform "Negative Superposition" from (8) and (2), yielding

$$b \not\approx b \quad (9)$$

Finally, we perform an "Equality Resolution" inference from (9), yielding the empty clause. No other inferences are possible.

(Note that the "Negative Superposition" inferences that produced (5) did not violate the condition "u is not a variable." It was t that was a variable.)

Exercise 12.2: Refute the following set of equational clauses by superposition:

$$f(x) \not\approx c \lor f(x) \approx b \quad (1)$$
$$f(f(x)) \approx x \quad (2)$$
$$b \not\approx c \quad (3)$$

Choose an appropriate ordering and perform only inferences that satisfy the ordering restrictions.

Proposed solution. We use the kbo with $f \succ c \succ b$ and weight 1 for all symbols and variables. From (2) and (1) we obtain by "Negative Superposition"

$$x \not\approx c \lor f(f(x)) \approx b \quad (4)$$

From (4) and (2) we obtain by "Positive Superposition"

$$x \not\approx c \lor b \approx x \quad (5)$$

From (5) we obtain by "Equality Resolution"

$$b \approx c \quad (6)$$

From (6) and (3) we obtain by "Negative Superposition"

$$b \not\approx b \quad (7)$$

From (7) we obtain by "Equality Resolution" the empty clause.

Exercise 12.3: Consider the following set of equational clauses:

$$f(b) \approx \mathit{true} \quad (1)$$
$$f(x) \not\approx \mathit{true} \lor f(g(x)) \approx \mathit{true} \quad (2)$$

(a) Saturate this set by computing superposition inferences *ignoring ordering restrictions.*

(b) Choose an appropriate ordering and perform only inferences that satisfy the ordering restrictions.

Proposed solution. (a) If we ignore the ordering restrictions, we can derive

$$true \not\approx true \lor f(g(b)) \approx true \quad (3)$$

via "Negative Superposition" from (1) and (2) and

$$f(g(b)) \approx true \quad (4)$$

via "Equality Resolution" from (3). Next, we cound continue and generate the clauses

$$\begin{array}{cr} true \not\approx true \lor f(g(g(b))) \approx true & (5) \\ f(g(g(b))) \approx true & (6) \\ true \not\approx true \lor f(g(g(g(b)))) \approx true & (7) \\ f(g(g(g(b)))) \approx true & (8) \\ \vdots & \end{array}$$

The limit of this process is an infinite saturated clause set.

(b) We assume an instance of lpo that assigns the lowest precedence to *true*. Then a "Negative Superposition" inference from (1) and (2) is impossible, because the lhs of the first literal of (2) is not maximal. Thus, the set consisting of (1) and (2) is saturated.

Exercise 12.4: Prove that the ground "Equality Resolution" inference rule is sound:

Equality Resolution: $\dfrac{C' \lor s \not\approx s}{C'}$

Proposed solution. We need to prove the entailment $C' \lor s \not\approx s' \models C'$. Let $\mathcal{A}$ be a model of the premise $C' \lor s \not\approx s$. We will show that $\mathcal{A}$ is a model of the conclusion C'. Since $s \not\approx s$ is false in any model, if the premise is true in $\mathcal{A}$, we need that C' is true in $\mathcal{A}$, as desired.

Exercise 12.5: Prove that the ground "Equality Factoring" inference rule is sound:

Equality Factoring: $\dfrac{C' \lor s \approx t' \lor s \approx t}{C' \lor t \not\approx t' \lor s \approx t'}$

Proposed solution. We need to prove the entailment $C' \lor s \approx t' \lor s \approx t \models C' \lor t \not\approx t' \lor s \approx t'$. Let $\mathcal{A}$ be a model of the premise $C' \lor s \approx t' \lor s \approx t$. We will show that the conclusion $C' \lor t \not\approx t' \lor s \approx t'$ is true in $\mathcal{A}$.

Since the premise is true in $\mathcal{A}$, this means that at least one of the three disjuncts is true in $\mathcal{A}$:

- If C' is true in $\mathcal{A}$, then the conclusion's first literal is true in $\mathcal{A}$ and hence the entire conclusion is true in $\mathcal{A}$.
- If $s \approx t'$ is true in $\mathcal{A}$, then the conclusion's third literal is true in $\mathcal{A}$ and hence the entire conclusion is true in $\mathcal{A}$.
- If $s \approx t$ is true in $\mathcal{A}$, we perform a case distinction on whether or not $t \not\approx t'$ is true in $\mathcal{A}$. In the first case, the conclusion's second disjunct is then true in $\mathcal{A}$ and hence the entire conclusion is true in $\mathcal{A}$. In the second case, $\mathcal{A}$ interprets t and t' as the same value, and since we know that $s \approx t$ is true in $\mathcal{A}$, then we also know that $s \approx t'$ is true in $\mathcal{A}$. Hence the entire conclusion is true in $\mathcal{A}$.

Exercise 12.6: Prove that the ground "Negative Superposition" inference rule is sound:

Neg. Superposition: $$\frac{D' \lor t \approx t' \qquad C' \lor s[t] \not\approx s'}{D' \lor C' \lor s[t'] \not\approx s'}$$

Proposed solution. We need to prove the entailment $D' \lor t \approx t'$, $C' \lor s[t] \not\approx s' \models D' \lor C' \lor s[t'] \not\approx s'$. Let $\mathcal{A}$ be a model of the premises $D' \lor t \approx t'$ and $C' \lor s[t] \not\approx s'$. We will show that the conclusion $D' \lor C' \lor s[t'] \not\approx s'$ is true in $\mathcal{A}$.

If either D' or C' is true in $\mathcal{A}$, then the conclusion is clearly true in $\mathcal{A}$. Otherwise, both D' and C' are false in $\mathcal{A}$. Since the first premise is true in $\mathcal{A}$, its literal $t \approx t'$ must be true, and since the second premise is true in $\mathcal{A}$, its literal $s[t] \not\approx s'$ must be true in $\mathcal{A}$. By congruence, $s[t] \not\approx s'$ is also true in $\mathcal{A}$, and hence the entire conclusion is true in $\mathcal{A}$.

The proof for ground "Positive Superposition" is analogous.

Exercise 12.7: In the lecture notes, it is stated that the ordering restrictions of the inference rules of the superposition calculus must be satisfied *after applying the mgu to the premises*. Give a simple example that shows that a literal may be maximal in a clause but that the maximality requirement may be violated after applying the mgu.

Proposed solution. We use the lpo with the precedence $c \succ b$. Consider a "Negative Superposition" inference with the premises $c \approx b \vee x \approx b$ and $b \not\approx b$. The mgu is $\sigma = \{x \mapsto b\}$. The literal $x \approx b$ is maximal in the first premise, but $(x \approx b)\sigma = b \approx b$ is not maximal in $(c \approx b \vee x \approx b)\sigma = c \approx b \vee b \approx b$.

Exercise 12.8 (∗)**:** Find a small unsatisfiable set N of equational clauses and a term ordering $\succ$ such that N is saturated w.r.t. the superposition calculus *excluding* the "Equality Factoring" rule and N does not contain $\bot$. The existence of such a set implies that the superposition calculus is incomplete without "Equality Factoring."

Hint: Recall the informal motivation for adding "Equality Factoring" to the calculus.

Proposed solution. We take $N := \{c \approx b,\ d \approx c \vee d \approx b,\ d \not\approx b\}$ and the lpo with the precedence $d \succ c \succ b$.

The set N is clearly unsatisfiable. For saturation, we must show that all inferences from N other than "Equality Factoring" inferences are redundant w.r.t. N. At first glance, a "Negative Superposition" inference appears to be possible from the second and third clauses; however, it would violate the ordering restriction that the left premise must not be larger than the right premise. Other inferences are clearly impossible.