# Automated Theorem Proving

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## Exercises 11: Completion

**Exercise 11.1:** Apply the Knuth–Bendix procedure to the set of equations

$$h(f(x)) \approx g(x) \qquad (1)$$
$$f(b) \approx c \qquad (2)$$

and transform it into a finite convergent term rewrite system. Use the Knuth–Bendix ordering with weight 1 for all function symbols and variables and the precedence $h \succ g \succ f \succ b \succ c$.

**Proposed solution.** We first apply "Orient" to (1) and (2), which results in the rewrite rules

$$h(f(x)) \rightarrow g(x) \qquad (3)$$
$$f(b) \rightarrow c \qquad (4)$$

Then we apply "Deduce" between (3) at position 1 and (4), yielding the equation

$$f(c) \approx g(b) \qquad (5)$$

Then we apply "Orient" to (5), which results in the rewrite rule

$$g(b) \rightarrow f(c) \qquad (6)$$

At this point, all critical pairs between persisting rules have been computed and all equations have been eliminated. We can stop now. Thus the set $\{(3), (4), (6)\}$ is a finite convergent term rewrite system.

**Exercise 11.2:** Let $E$ be the following set of equations over $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0\}, \emptyset)$.

$$f(g(x)) \approx h(x) \qquad (1)$$
$$g(f(x)) \approx h(x) \qquad (2)$$

Apply the Knuth–Bendix completion procedure to $E$ and transform it into a finite convergent term rewrite system. Use a Knuth–Bendix ordering with weight 1 for all function symbols and variables and the precedence $f \succ g \succ h \succ b$. Use a reasonable strategy.

**Proposed solution.** We start with the given equations (1)–(2).

$$\begin{array}{rlrl}
f(g(x)) \approx h(x) & (1) & g(f(x)) \approx h(x) & (2) \\
f(g(x)) \rightarrow h(x) & (3) & g(f(x)) \rightarrow h(x) & (4) \\
h(f(x)) \approx f(h(x)) & (5) & f(h(x)) \rightarrow h(f(x)) & (6) \\
h(g(x)) \approx g(h(x)) & (7) & g(h(x)) \rightarrow h(g(x)) & (8) \\
h(h(x)) \approx g(h(f(x))) & (9) & h(h(x)) \approx h(g(f(x))) & (10) \\
h(h(x)) \approx h(h(x)) & (11) & h(h(x)) \approx f(h(g(x))) & (12) \\
h(h(x)) \approx h(f(g(x))) & (13) & h(h(x)) \approx h(h(x)) & (14)
\end{array}$$

By applying "Orient" twice, we replace (1)–(2) by the corresponding rewrite rules (3)–(4).

Using the critical pair between rules (3) and (4), "Deduce" adds equation (5). Then "Orient" replaces equation (5) by rule (6).

Using the critical pair between rules (4) and (3), "Deduce" adds equation (7). Then "Orient" replaces equation (7) by rule (8).

Using the critical pair between rules (4) and (6), "Deduce" adds equation (9). "Simplify-Eq" uses rewrite rule (8) to replace equation (9) by equation (10); then "Simplify-Eq" uses rewrite rule (4) to replace equation (10) by equation (11). Equation (11) is trivial, so it can be eliminated using "Delete."

Using the critical pair between rules (3) and (8), "Deduce" adds equation (12). "Simplify-Eq" uses rewrite rule (6) to replace equation (12) by equation (13); then "Simplify-Eq" uses rewrite rule (3) to replace equation (13) by equation (14). Equation (14) is trivial, so it can be eliminated using "Delete."

Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now. The final rewrite system is $\{(3), (4), (6), (8)\}$.

**Exercise 11.3:** Let $E$ be the following set of equations over $\Sigma = (\{f/2,\ g/1,\ b/0,\ c/0\}, \emptyset)$.

$$\begin{aligned} f(x,x) &\approx f(x,b) && (1)\\ f(x,x) &\approx f(c,x) && (2)\\ f(x,x) &\approx g(x) && (3) \end{aligned}$$

Apply the Knuth–Bendix completion procedure to $E$ and transform it into a finite convergent term rewrite system. Use a Knuth–Bendix ordering with weight 1 for all function symbols and variables and the precedence $f \succ g \succ b \succ c$. Use a reasonable strategy.

**Proposed solution.** We start with the given equations (1)–(3).

$$\begin{array}{rlcrl} f(x,x) \approx f(x,b) & (1) & & f(x,x) \approx f(c,x) & (2)\\ f(x,x) \approx g(x) & (3) & & f(x,x) \to g(x) & (4)\\ g(x) \approx f(x,b) & (5) & & g(x) \approx f(c,x) & (6)\\ f(x,b) \to g(x) & (7) & & f(c,x) \to g(x) & (8)\\ g(b) \approx g(b) & (9) & & g(c) \approx g(c) & (10)\\ g(b) \approx g(c) & (11) & & g(b) \to g(c) & (12) \end{array}$$

Equations (1) and (2) cannot be oriented, so we apply "Orient" to replace (3) by (4). Now we can use "Simplify-Eq" twice with (4) to replace equation (1) by (5) and to replace equation (2) by (6). By applying "Orient" twice, we replace (5) and (6) by the corresponding rewrite rules (7) and (8). Using the critical pair between rules (4) and (7), the "Deduce" rule adds equation (9), which is trivial and gets eliminated using "Delete." Using the critical pair between rules (4) and (8), the "Deduce" rule adds equation (10), which is also trivial and gets eliminated using "Delete." Finally, using the critical pair between rules (7) and (8), the "Deduce" rule adds equation (11); this equation is replaced by the corresponding rewrite rule (12) using "Orient." Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now. The final rewrite system is $\{(4), (7), (8), (12)\}$.

**Exercise 11.4:** Let $E$ be the following set of equations over $\Sigma = (\{f/1,\ g/1,\ h/1\}, \emptyset)$.

$$\begin{aligned} f(g(f(x))) &\approx h(x) && (1)\\ g(h(x)) &\approx x && (2) \end{aligned}$$

Apply the Knuth–Bendix completion procedure to $E$ and transform it into a finite convergent term rewrite system. Use the Knuth–Bendix ordering with weight 1 for all function symbols and variables and the precedence $f \succ g \succ h$. Use a reasonable strategy.

**Proposed solution.** We start with the given equations (1)–(2).

| | | | |
|---|---|---|---|
| $f(g(f(x))) \approx h(x)$ | (1) | $g(h(x)) \approx x$ | (2) |
| $f(g(f(x))) \to h(x)$ | (3) | $g(h(x)) \to x$ | (4) |
| $h(g(f(x))) \approx f(g(h(x)))$ | (5) | $h(g(f(x))) \approx f(x)$ | (6) |
| $h(g(f(x))) \to f(x)$ | (7) | $f(g(f(x))) \approx h(g(h(x)))$ | (8) |
| $h(x) \approx h(g(h(x)))$ | (9) | $h(x) \approx h(x)$ | (10) |
| $g(f(x)) \approx g(f(x))$ | (11) | | |

By applying "Orient" twice, we replace (1)–(2) by the corresponding rewrite rules (3)–(4).

Using the critical pair between rule (3) and a renamed copy of itself, the "Deduce" rule adds equation (5). The "Simplify-Eq" rule uses rewrite rule (4) to replace equation (5) by equation (6). The "Orient" rule replaces equation (6) by rule (7).

Using the critical pair between rules (7) and (3), the "Deduce" rule adds equation (8). The "Simplify-Eq" rule uses rewrite rule (3) to replace equation (8) by equation (9). The "Simplify-Eq" rule uses rewrite rule (4) to replace equation (9) by equation (10). Equation (10) is trivial, so it can be eliminated using "Delete."

Using the critical pair between rules (4) and (7), the "Deduce" rule adds equation (11). Equation (11) is again trivial, so it can be eliminated using "Delete."

Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now. The final rewrite system is $\{(3), (4), (7)\}$.

**Exercise 11.5:** Let $E$ be the following set of equations over $\Sigma = (\{f/2,\ g/1,\ h/1,\ b/0\}, \emptyset)$.

$$f(g(x), x) \approx b \qquad (1)$$
$$f(x, b) \approx x \qquad (2)$$
$$g(h(x)) \approx x \qquad (3)$$

Apply the Knuth–Bendix completion procedure to $E$ and transform it into a finite convergent term rewrite system. Use a Knuth–Bendix ordering with weight 1 for all function symbols and variables and the precedence $f \succ g \succ h \succ b$. Use a reasonable strategy.

**Proposed solution.** We start with the three given equations (1)–(3)

$$
\begin{array}{rlrl}
f(g(x),x) \approx b & (1) & f(g(x),x) \to b & (4) \\
f(x,b) \approx x & (2) & f(x,b) \to x & (5) \\
g(h(x)) \approx x & (3) & g(h(x)) \to x & (6) \\
b \approx g(b) & (7) & g(b) \to b & (8) \\
b \approx f(x,h(x)) & (9) & f(x,h(x)) \to b & (10) \\
b \approx f(b,b) & (11) & & \\
b \approx b & (12) & &
\end{array}
$$

By applying "Orient" three times, we replace (1)–(3) by the corresponding rewrite rules (4)–(6). Using the critical pair between rules (4) and (5), the "Deduce" rule adds equation (7). The "Orient" rule replaces equation (7) by rule (8). Using the critical pair between rules (4) and (6), the "Deduce" rule adds equation (9). The "Orient" rule replaces equation (9) by rule (10). Using the critical pair between rules (4) and (8), the "Deduce" rule adds equation (11). The "Simplify-Eq" rule uses the rewrite rule (5) to replace equation (11) by equation (12). Equation (12) is trivial, so it can be eliminated using "Delete." Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now. The final rewrite system is $\{(4),(5),(6),(8),(10)\}$.

**Exercise 11.6:** Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/4,\ b/0,\ c/0,\ d/0,\ e/0\}$. Let $\succ$ be the lpo with the precedence $f \succ b \succ c \succ d \succ e$. Let $E$ be the set of equations

$$
\begin{array}{ll}
f(w,x,y,z) \approx f(x,y,z,w) & (1) \\
f(c,d,e,b) \approx b & (2) \\
f(c,b,e,d) \approx c & (3)
\end{array}
$$

Compute the set of semicritical pairs $\mathrm{SC}_\succ(E)$.

**Proposed solution.** There are three semicritical pairs:

Between (1) at position $\varepsilon$ of the lhs and the rhs of a renamed copy of (1):
mgu $\{w \mapsto x',\ x \mapsto y',\ y \mapsto z',\ z \mapsto w'\}$,
$f(y',z',w',x') \leftarrow f(x',y',z',w') \to f(w',x',y',z')$,
semicritical pair: $\langle f(y',z',w',x'), f(w',x',y',z')\rangle$.

Between (2) at position $\varepsilon$ of the lhs and the rhs of (1):
mgu $\{x \mapsto c,\ y \mapsto d,\ z \mapsto e,\ w \mapsto b\}$,
$b \leftarrow f(c,d,e,b) \to f(b,c,d,e)$,
semicritical pair: $\langle b, f(b,c,d,e)\rangle$.

Between (3) at position $\varepsilon$ of the lhs and the lhs of (1):
mgu $\{w \mapsto c, x \mapsto b, y \mapsto e, z \mapsto d\}$,
$c \leftarrow f(c, b, e, d) \rightarrow f(b, e, d, c)$,
semicritical pair: $\langle c, f(b, e, d, c)\rangle$.

**Exercise 11.7:** Use unfailing completion to transform the set of equations

$$b \approx c \qquad (1)$$
$$b + d \approx e \qquad (2)$$
$$x + y \approx y + x \qquad (3)$$

into a ground convergent set of equations. Use the lpo with the precedence $+ \succ b \succ c \succ d \succ e$.

**Proposed solution.** We first apply "Orient" to (1) and (2), resulting in

$$b \rightarrow c \qquad (4)$$
$$b + d \rightarrow e \qquad (5)$$

Then we use "L-Simplify-Rule" on (5) and (4) to replace (5) with

$$c + d \rightarrow e \qquad (6)$$

Then we use "Deduce" on (3) and (6) to produce

$$d + c \approx e \qquad (7)$$

Then we use "Orient" on (7), yielding

$$d + c \rightarrow e \qquad (8)$$

At this point, all semicritical pairs between persisting equations and rules have been computed. We can stop now. Thus the set $\{(3), (4), (6), (8)\}$ is ground convergent.