Automated Theorem Proving

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Exercises 10: Termination

Exercise 10.1: Let $\Sigma = (\{f/1, g/2, h/1, b/0, c/0\}, \emptyset)$ and let

$$t_{1} = g(h(x), h(c))$$

$$t_{2} = g(x, x),$$

$$t_{3} = g(b, f(x)),$$

$$t_{4} = f(g(x, y)),$$

$$t_{5} = h(g(x, c)).$$

Determine for each $1 \le i < j \le 5$ whether t_i and t_j are uncomparable or comparable (and if so, which term is larger) with respect to

- (a) the lexicographic path ordering with the precedence $f \succ g \succ h \succ b \succ c$,
- (b) the Knuth–Bendix ordering with the precedence $h \succ f \succ g \succ b \succ c$, where h has weight 0, b has weight 3, and all other symbols and variables have weight 1,
- (c) the polynomial ordering over $\{n \in \mathbb{N} \mid n \ge 1\}$ with $P_f(X_1) = X_1 + 1$, $P_g(X_1, X_2) = 2X_1 + X_2 + 1$, $P_h(X_1) = 3X_1$, $P_b = 1$ and $P_c = 3$.

Proposed solution. (a)

 $\begin{array}{l} t_1 \succ t_2 \\ t_1 \prec t_3 \\ t_1 \prec t_4 \\ t_1 \succ t_5 \\ t_2 \text{ and } t_3 \text{ are uncomparable} \\ t_2 \prec t_4 \\ t_2 \text{ and } t_5 \text{ are uncomparable} \\ t_3 \prec t_4 \end{array}$

 t_3 and t_5 are uncomparable $t_4 \succ t_5$ (b) t_1 and t_2 are uncomparable $t_1 \prec t_3$ $t_1 \prec t_4$ $t_1 \prec t_5$ t_2 and t_3 are uncomparable t_2 and t_4 are uncomparable t_2 and t_5 are uncomparable t_3 and t_4 are uncomparable $t_3 \succ t_5$ $t_4 \succ t_5$ (c) $P_{t_1}(X) = 6X + 10$ $P_{t_2}(X) = 3X + 1$ $P_{t_3}(X) = X + 4$ $P_{t_4}(X,Y) = 2X + Y + 2$ $P_{t_5}(X) = 6X + 12$ $t_1 \succ t_2$ $t_1 \succ t_3$ t_1 and t_4 are uncomparable $t_1 \prec t_5$ t_2 and t_3 are uncomparable t_2 and t_4 are uncomparable $t_2 \prec t_5$ $t_3 \prec t_4$ $t_3 \prec t_5$ t_4 and t_4 are uncomparable

Exercise 10.2: Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let \succ be a strict partial ordering on Ω , and let $s, t \in T_{\Sigma}(X)$.

(a) Prove: If s contains a subterm $s' = f(s_1, \ldots, s_n)$ such that $var(s') \supseteq var(t)$ and $f \succ g$ for all function symbols g occurring in t, then $s \succ_{\text{lpo}} t$.

(b) Refute: If s contains a subterm $s' = f(s_1, \ldots, s_n)$ such that $\operatorname{var}(s) \supseteq \operatorname{var}(t)$ and $f \succ g$ for all function symbols g occurring in t, then $s \succ_{\operatorname{lpo}} t$.

Proposed solution. (a) Since every lpo is a simplification ordering, we know that $s \succeq_{\text{lpo}} s'$. Therefore it suffices to show that $s' \succ_{\text{lpo}} t$. We prove this by induction over the structure of t:

If t is a variable, then by assumption $t \in var(t) \subseteq var(s')$; so $s' \succ_{lpo} t$ by case (1) of the lpo definition.

Otherwise $t = g(t_1, \ldots, t_m)$ with $f \succ g$. By induction, $s' \succ_{\text{lpo}} t_j$ for all $1 \le j \le m$. So $s' \succ_{\text{lpo}} t$ by case (2)(b) of the lpo definition.

(b) Choose the precedence $f \succ g \succ h$, then $s = h(f(x), y) \not\succ_{\text{lpo}} g(y) = t$, but $f \succ g$ and $\operatorname{var}(s) \supseteq \operatorname{var}(t)$.

Exercise 10.3: Determine for each of the following statements whether they are true or false:

- (1) $f(g(x)) \succ f(x)$ in every simplification ordering \succ .
- (2) $f(f(x)) \succ f(y)$ in every simplification ordering \succ .
- (3) If \succ is an lpo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.
- (4) If \succ is a kbo, then $f(x) \succ g(x)$ implies $f(x) \succ g(g(x))$.
- (5) If \succ is an lpo, then $h(f(x), y, y) \succ h(x, z, z)$.
- (6) If \succ is a kbo, then $h(f(x), f(y), z) \succ h(x, f(z), y)$.
- (7) There is a reduction ordering \succ such that $f(x) \succ g(f(x))$.
- (8) There is a reduction ordering \succ such that $f(f(x)) \succ f(g(f(x)))$.

Proposed solution. (1) True. $g(x) \succ x$ by the subterm property, so $f(g(x)) \succ f(x)$ by compatibility with contexts.

- (2) False. If $t \succ t'$ in a simplification ordering, then $\operatorname{var}(t') \subseteq \operatorname{var}(t)$.
- (3) True. In an lpo, $f(x) \succ g(x)$ implies $f \succ g$, hence $f(x) \succ g(g(x))$.
- (4) False. Choose weight 3 for f and weight 2 for g.
- (5) False. If $t \succ t'$ in a simplification ordering, then $\operatorname{var}(t') \subseteq \operatorname{var}(t)$.

(6) True. The variable condition for the kbo is satisfied. If f has a positive weight, then the weight of the first term is larger than the weight of the second term; if f has weight 0, then both terms have the same weight, and since $f(x) \succ x$, the argument tuple of the first term is lexicographically larger than the argument tuple of the second term.

(7) False. Otherwise we have $f(x) \succ g(f(x)) \succ g(g(f(x))) \succ g(g(g(f(x)))) \succ \cdots$, contradicting well-foundedness.

(8) True. The rewrite system $R = \{f(f(x)) \to f(g(f(x)))\}$ is terminating, so \to_R^+ is a reduction ordering with the desired property.

Exercise 10.4 (*): Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let *h* be a unary function symbol in Ω , and let \succ be precedence on Ω such that *h* is the smallest element of Ω w.r.t. \succ .

Prove: For all terms $s, t \in T_{\Sigma}(X)$, we have $s \succ_{\text{lpo}} t$ if and only if $s \succeq_{\text{lpo}} h(t)$.

Proposed solution. Since an lpo is a simplification ordering, we know that $h(t) \succ_{\text{lpo}} t$, so $s \succeq_{\text{lpo}} h(t)$ implies $s \succeq_{\text{lpo}} h(t) \succ_{\text{lpo}} t$ and thus $s \succ_{\text{lpo}} t$ by transitivity. This proves the "if" part.

The "only if" part is proved by induction over |s| + |t|. First assume that the top symbol f of s is different from h. Since h is the smallest element of the precedence, we have $f \succ h$, so $s \succ_{\text{lpo}} t$ implies $s \succ_{\text{lpo}} h(t)$ by Case (2b).

Otherwise s = h(s').

If $h(s') \succ_{\text{lpo}} t$ by Case (1), then t is a variable that occurs in h(s'). So either s' = t, then s = h(s') = h(t), or $s' \succ t$, then $s' \succ_{\text{lpo}} t$, and by compatibility with contexts $s = h(s') \succ_{\text{lpo}} h(t)$.

If $h(s') \succ_{\text{lpo}} t$ by Case (2a), then s' = t or $s' \succ_{\text{lpo}} t$. In the first case s = h(s') = h(t), in the second case $s = h(s') \succ_{\text{lpo}} h(t)$ by compatibility with contexts.

We cannot have $h(s') \succ_{\text{lpo}} t$ by Case (2b) since h is the smallest element of the precedence. So it remains to consider the case that $h(s') \succ_{\text{lpo}} t$ by Case (2c). Then t = h(t')and $s' \succ_{\text{lpo}} t'$. By induction, we get $s' \succeq_{\text{lpo}} h(t')$. So s' = h(t') or $s' \succ_{\text{lpo}} h(t')$, therefore s = h(s') = h(h(t')) = h(t) or $s = h(s') \succ_{\text{lpo}} h(h(t')) = h(t)$ by Case (2c).

Exercise 10.5: Let $\Sigma = (\{f/2, g/2, h/2\}, \emptyset)$. Let R be the term rewrite system

$$\{ g(x, f(x, y)) \to h(y, g(x, y)), \quad h(x, y) \to g(y, y) \}$$

Is there a lexicographic path ordering \succ_{lpo} such that $\rightarrow_R \subseteq \succ_{\text{lpo}}$? If yes, give the precedence of this lpo; if no, explain why such an lpo does not exist.

Proposed solution. \rightarrow_R is contained in an loo with the precedence $f \succ h \succ g$.

Exercise 10.6: Let $\Sigma = (\{f/2, g/1, h/1, b/0\}, \emptyset)$. Let R be the term rewrite system

$$\{ f(g(x), y) \to g(f(x, x)), \quad h(f(x, b)) \to g(x) \} \}$$

Is there a Knuth–Bendix ordering \succ_{kbo} such that $\rightarrow_R \subseteq \succ_{\text{kbo}}$? If yes, give the weights and precedence of this kbo; if no, explain why such a kbo does not exist.

Proposed solution. \rightarrow_R is not contained in any kbo, since the first rewrite rule has more occurrences of x in the right-hand side than in the left-hand side.

Exercise 10.7: Let $\Sigma = (\{f/1, g/1, b/0, c/0\}, \emptyset)$. Let R be the term rewrite system

 $\{ f(g(x)) \to g(g(f(x))), \quad c \to f(b) \}$

Is there a polynomial ordering $\succ_{\mathcal{A}}$ in which the function symbols are interpreted by linear polynomials over $U_{\mathcal{A}} = \{n \in \mathbb{N} \mid n \geq 1\}$ such that $\rightarrow_R \subseteq \succ_{\mathcal{A}}$? If yes, give the polynomials by which the symbols of Σ are interpreted; if no, explain why such an ordering does not exist.

Proposed solution. \rightarrow_R is contained in a polynomial ordering where the symbols in Σ are interpreted by $P_f(X_1) = 3X_1$, $P_q(X_2) = X_2 + 1$, $P_b = 1$, $P_c = 4$.

Exercise 10.8: Let $\Sigma = (\Omega, \emptyset)$ be a finite signature. For $t \in T_{\Sigma}(X)$ we define depth $(t) = \max\{|p| \mid p \in \text{pos}(t)\}$. Let \succ be a strict partial ordering on Ω . The binary relation \succ_{do} on $T_{\Sigma}(X)$ is defined by: $s \succ_{\text{do}} t$ if and only if

- (1) $\#(x,s) \ge \#(x,t)$ for all variables x and depth(s) > depth(t), or
- (2) $\#(x,s) \ge \#(x,t)$ for all variables x, depth(s) = depth(t), and

(a)
$$s = f(s_1, ..., s_m), t = g(t_1, ..., t_n), \text{ and } f \succ g, \text{ or } s_1 \in [0, 1]$$

(b) $s = f(s_1, \dots, s_m), t = f(t_1, \dots, t_m)$, and $(s_1, \dots, s_m) (\succ_{do})_{lex} (t_1, \dots, t_m).$

Give an example that shows that $>_{do}$ is not a reduction ordering.

Proposed solution. The relation \succ_{do} is not stable under substitutions. For instance, we have $s = f(x, g(g(y))) \succ_{do} f(g(x), g(y)) = t$ since depth(s) = 3 and depth(t) = 2, but if $\sigma = \{x \mapsto h(h(z))\}$, then we get $s\sigma = f(h(h(z)), g(g(y))) \prec_{do} f(g(h(h(z))), g(y)) = t\sigma$ since depth $(s\sigma) = 3$ and depth $(t\sigma) = 4$.

Exercise 10.9 (*): Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let \succ be a simplification ordering. Let R be a TRS over $T_{\Sigma}(X)$ such that $l \succ r$ for all $l \rightarrow r \in R$. Let h be an n-ary function symbol in Ω (with n > 0) that does not occur in any left-hand side of a rule in R. Prove: If R is confluent, then $R \cup \{h(x, \ldots, x) \rightarrow x\}$ is confluent.

Proposed solution. First, we observe that $h(x, \ldots, x)$ is larger than its proper subterm x in every simplification ordering \succ . Therefore $l \succ r$ holds in fact for all $l \rightarrow r \in R \cup \{h(x, \ldots, x) \rightarrow x\}$. Consequently, $R \cup \{h(x, \ldots, x) \rightarrow x\}$ is terminating.

Second, we observe that the rewrite rule $h(x, \ldots, x) \to x$ has neither a critical pair with itself, nor with any rule $l \to r \in R$ (since h does not occur in l). Consequently, every critical pair between rules in $R \cup \{h(x, \ldots, x) \to x\}$ is a critical pair between rules in R. Since R is confluent, all critical pairs between rules in R are joinable in R, and hence also joinable in $R \cup \{h(x, \ldots, x) \to x\}$.

Using the critical pair theorem, we conclude that $R \cup \{h(x, \ldots, x) \to x\}$ is locally confluent; and since it is terminating, it is also confluent.

Exercise 10.10: Let $\Sigma = (\{f/1, g/2, h/2, b/0, c/0\}, \emptyset)$. Let *E* be the following set of equations over Σ :

$$f(f(x)) \approx g(b, x) \tag{1}$$

$$h(f(y), y') \approx f(h(y, y')) \tag{2}$$

$$g(h(z, z), c) \approx h(z, b) \tag{3}$$

(a) Suppose that the three equations in E are turned into rewrite rules by orienting them from left to right. Give all critical pairs between the resulting three rules.

(b) It is possible to orient the equations in E using an appropriate kbo so that there are no critical pairs between the resulting rules. Give the weights and precedence of the kbo, and explain how the equations are oriented.

Proposed solution. (a) There are three critical pairs:

Between (1) at position 1 and a renamed copy of (1): mgu $\{x \mapsto f(x')\},\$ $g(b, f(x')) \leftarrow f(f(f(x'))) \rightarrow f(g(b, x')),\$ critical pair: $\langle g(b, f(x')), f(g(b, x')) \rangle.$

Between (2) at position 1 and (1): mgu $\{y \mapsto f(x)\},\$ $f(h(f(x), y')) \leftarrow h(f(f(x)), y') \rightarrow h(g(b, x), y'),\$ critical pair: $\langle f(h(f(x), y')), h(g(b, x), y') \rangle.$ Between (3) at position 1 and (2): mgu $\{y' \mapsto f(y), z \mapsto f(y)\},$ $h(f(y), b) \leftarrow g(h(f(y), f(y)), c) \rightarrow g(f(h(y, f(y))), c),$ critical pair: $\langle h(f(y), b), g(f(h(y, f(y))), c) \rangle.$

(b) To avoid a critical pair between (1) and itself, we must orient (1) right to left. To avoid a critical pair between (3) and (2), we must orient (2) right to left and (3) left to right.

There are several possibilities to orient the equations in this way using a kbo, for instance by choosing weight 1 for all function symbols and variables and a precedence $g \succ f \succ h \succ b \succ c$.

Exercise 10.11: Let $\Sigma = (\{f/1, g/1, h/1, b/0, c/0\}, \{P/2, Q/1, R/2\})$. Let N be the following set of clauses over Σ :

$$P(f(x), x) \lor P(c, x) \lor R(g(x), x)$$
(1)

$$\neg P(y, f(y))$$
(2)

$$\neg P(y, c) \lor \neg P(z, h(y)) \lor Q(z)$$
(3)

$$Q(b) \lor Q(x) \lor \neg R(g(x), x)$$
(4)

$$R(g(c), y)$$
(5)

(a) Suppose that the atom ordering \succ is a lexicographic path ordering with the precedence $P \succ Q \succ R \succ f \succ g \succ h \succ b \succ c$ and that the selection function *sel* selects no literals. Compute all $\operatorname{Res}_{sel}^{\succ}$ inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the conclusions of the inferences computed in part (a) is redundant w.r.t. N. Which one? Why?

Proposed solution. (a) In (1), P(c, x) and R(g(x), x) are not maximal since $P(f(x), x) \succ P(c, x)$ and $P(f(x), x) \succ R(g(x), x)$. In (3), Q(z) is not maximal since $\neg P(z, h(y)) \succ Q(z)$. In (4), $\neg R(g(x), x)$ is not maximal since $Q(x) \succ \neg R(g(x), x)$. The remaining literals are maximal in their clauses: (1)1, (2)1, (3)1, (3)2, (4)1, (4)2, (5)1. This yields the following three inferences:

Res. (1)1, (3)1: mgu: $\{x \mapsto c, y \mapsto f(c)\}$ $P(c,c) \lor R(g(c),c) \lor \neg P(z,h(f(c))) \lor Q(z)$ Res. (1)1, (3)2: mgu: $\{x \mapsto h(y), z \mapsto f(h(y))\}$ $P(c,h(y)) \lor R(g(h(y)),h(y)) \lor \neg P(y,c) \lor Q(f(h(y)))$ Fact. (4)1, (4)2: mgu: $\{x \mapsto b\}$

 $Q(b) \lor \neg R(g(b), b)$

(b) The conclusion of the first inference above contains the subclause R(g(c), c), which is an instance of clause (5). Therefore, every ground instance of the conclusion follows from a smaller ground instance of (5). Hence the conclusion is redundant.

Exercise 10.12: Let $\Sigma = (\{f/1, g/1, h/1, b/0, c/0\}, \{P/2, Q/1, R/2\})$. Let N be the following set of clauses over Σ :

$$P(x, f(x)) \lor P(x, x) \tag{1}$$

$$\neg P(h(z), x) \lor \neg P(y, f(f(x))) \lor \neg Q(x) \lor Q(f(x))$$
(2)

$$\neg Q(h(f(x))) \lor R(h(b), y) \tag{3}$$

$$\neg R(y, g(c)) \lor Q(g(x)) \tag{4}$$

$$\neg Q(h(y)) \tag{5}$$

(a) Suppose that the atom ordering \succ is a Knuth–Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ Q \succ R \succ$ $f \succ g \succ h \succ b \succ c$ and that the selection function *sel* selects no literals. Compute all Res^{\succ}_{*sel*} inferences between the clauses (1)–(5). Do not compute inferences between derived clauses.

(b) One of the clauses (1)–(5) is redundant with respect to the others. Which one? Why? Give a brief explanation.

Proposed solution. (a) The following literals are maximal in clauses (1)-(5):

(1): literal 1 (literal 2 is smaller than literal 1);

(2): literals 1 and 2 (literals 3 and 4 are smaller than literal 2);

(3): literals 1 and 2;

(4): literals 1 and 2;

(5): literal 1.

From these, we get the following $\operatorname{Res}_{sel}^{\succ}$ inferences:

Resolution (1) literal 1 and (2) literal 1 (after renaming x in clause (2) to x'): mgu $\{x \mapsto h(z), x' \mapsto f(h(z))\},$ conclusion $P(h(z), h(z)) \lor \neg P(y, f(f(f(h(z))))) \lor \neg Q(f(h(z))) \lor Q(f(f(h(z))))).$

Resolution (1) literal 1 and (2) literal 2 (after renaming x in clause (2) to x'): mgu $\{x \mapsto f(x'), y \mapsto f(x')\},\$ conclusion $P(f(x'), f(x')) \lor \neg P(h(z), x') \lor \neg Q(x') \lor Q(f(x')).$

Resolution (3) literal 2 and (4) literal 1 (after renaming x and y in clause (4) to x' and y'):

 $\begin{array}{l} \mathrm{mgu} \; \{y' \mapsto h(b), \; y \mapsto g(c)\}, \\ \mathrm{conclusion} \; \neg Q(h(f(x))) \; \lor \; Q(g(x')). \end{array}$

(b) Clause (3) is subsumed by clause (5): After applying $\sigma = \{y \mapsto f(x)\}$ to (5), the literals of (5) are a proper submultiset of the literals of (3). By Prop. 3.13.1, this means that clause (3) is redundant (i.e., every ground instance $\neg Q(h(f(t))) \lor R(h(b), t')$ of (3) is implied by a smaller ground instance $\neg Q(h(f(t)))$ of (5)).

Exercise 10.13: Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{f/1, b/0, c/0\}$ and $\Pi = \{P/1\}$. Suppose that the atom ordering \succ is a Knuth–Bendix ordering with weight 1 for all predicate symbols, function symbols, and variables, and with the precedence $P \succ f \succ b \succ c$. Let $N = \{C_1, C_2, C_3\}$ with

$$C_1 = P(b)$$

$$C_2 = \neg P(f(f(c)))$$

$$C_3 = P(x) \lor P(f(x))$$

(a) Sketch what the set $G_{\Sigma}(N)$ of all ground instances of clauses in N looks like. How is it ordered with respect to the clause ordering $\succ_{\rm C}$?

(b) Construct the candidate interpretation $I_{G_{\Sigma}(N)}^{\succ}$ of the set of all ground instances of clauses in N. Which clauses in $G_{\Sigma}(N)$ are productive and what do they produce?

Proposed solution. (a) $G_{\Sigma}(N) = \{P(b)\} \cup \{\neg P(f(f(c)))\} \cup \{P(f^n(b)) \lor P(f^{n+1}(b)) \mid n \ge 0\} \cup \{P(f^n(c)) \lor P(f^{n+1}(c)) \mid n \ge 0\}.$

The clause ordering $\succ_{\mathcal{C}}$ orders $G_{\Sigma}(N)$ in the following way:

$$P(b)$$

$$\prec_{c} P(c) \lor P(f(c))$$

$$\prec_{c} P(b) \lor P(f(b))$$

$$\prec_{c} P(f(c)) \lor P(f^{2}(c))$$

$$\prec_{c} P(f(c)) \lor P(f^{2}(c))$$

$$\prec_{c} P(f^{2}(c)) \lor P(f^{2}(b))$$

$$\prec_{c} P(f^{2}(c)) \lor P(f^{3}(c))$$

$$\prec_{c} P(f^{2}(b)) \lor P(f^{3}(b))$$

$$\prec_{c} P(f^{3}(c)) \lor P(f^{4}(c))$$

$$\prec_{c} P(f^{3}(b)) \lor P(f^{4}(b))$$

$$\prec_{c} P(f^{4}(c)) \lor P(f^{5}(c))$$

$$\prec_{c} P(f^{4}(b)) \lor P(f^{5}(b))$$

$$\prec_{c} \vdots$$

Iter.	Clause C	R_C	E_C
0	P(b)	Ø	$\{P(b)\}$
1	$P(c) \lor P(f(c))$	$\{P(b)\}$	$\{P(f(c))\}$
2	$P(b) \lor P(f(b))$	$\{P(b), P(f(c))\}$	Ø
3	$P(f(c)) \lor P(f^2(c))$	$\{P(b), P(f(c))\}$	Ø
4	$\neg P(f^2(c))$	$\{P(b), P(f(c))\}$	Ø
5	$P(f(b)) \lor P(f^2(b))$	$\{P(b), P(f(c))\}$	$\{P(f^2(b))\}$
6	$P(f^2(c)) \lor P(f^3(c))$	$\{P(b), P(f(c)), P(f^2(b))\}$	$\{P(f^3(c))\}$
7	$P(f^2(b)) \vee P(f^3(b))$	$\{P(b), P(f(c)), P(f^{2}(b)), P(f^{3}(c))\}$	Ø
8	$P(f^3(c)) \vee P(f^4(c))$	$\{P(b), P(f(c)), P(f^{2}(b)), P(f^{3}(c))\}$	Ø
9	$P(f^3(b)) \vee P(f^4(b))$	$\{P(b), P(f(c)), P(f^{2}(b)), P(f^{3}(c))\}$	$\{P(f^4(b))\}$
10	$P(f^4(c)) \lor P(f^5(c))$	$\{P(b), P(f(c)), P(f^{2}(b)), P(f^{3}(c)), P(f^{4}(b))\}$	$\{P(f^5(c))\}$
11	$P(f^4(b)) \vee P(f^5(b))$	$\{P(b), P(f(c)), P(f^{2}(b)), P(f^{3}(c)), P(f^{4}(b)), P(f^{5}(c))\}$	Ø
:	:	:	:
•	•	•	•

(b) The following table summarizes the candidate interpretation construction:

The candidate interpretation $I_{G_{\Sigma}(N)}^{\succ}$ is $\{P(f^n(b)) \mid n \text{ is even}\} \cup \{P(f^n(c)) \mid n \text{ is odd}\}.$

Exercise 10.14 (*): Let $\Sigma = (\{f/1, b/0, c/0\}, \{P/1\})$. Let N be the following set of Σ -clauses:

$$P(b) (1) P(f(c)) (2) \neg P(x) \lor P(f(x)) (3)$$

Let \succ be a Knuth-Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P \succ f \succ b \succ c$. The ordering is extended to ground literals and ground clauses as usual. Give the smallest nonempty ground Σ -clauses C_1, C_2, C_3, C_4 such that

- (a) $C_1 \in G_{\Sigma}(N)$ and $C_1 \in \operatorname{Red}(N)$,
- (b) $C_2 \in G_{\Sigma}(N)$ and $C_2 \notin \operatorname{Red}(N)$,
- (c) $C_3 \notin G_{\Sigma}(N)$ and $C_3 \in \operatorname{Red}(N)$,
- (d) $C_4 \notin G_{\Sigma}(N)$ and $C_4 \notin \text{Red}(N)$.

Proposed solution. $C_1 = \neg P(c) \lor P(f(c))$ is a ground instance of (3) and it is entailed by (2), which is smaller than C_1 . In fact, C_1 is the only ground instance of a clause in N that is redundant w.r.t. N.

 $C_2 = P(b)$ is the smallest ground instance of a clause in N, namely (1). It is not entailed by smaller ground instances and therefore not redundant.

 $C_3 = \neg P(c) \lor P(c)$ is a tautology and the smallest redundant clause w.r.t. N.

 $C_4 = P(c)$ is the smallest nonempty Σ -clause. It is neither a ground instance of a clause in N nor entailed by smaller ground instances (and therefore not redundant).