Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 8: Semantic Tableaux

Exercise 8.1: Show unsatisfiability of the set of formulas

$$P \rightarrow (Q \rightarrow R) \qquad (1)$$

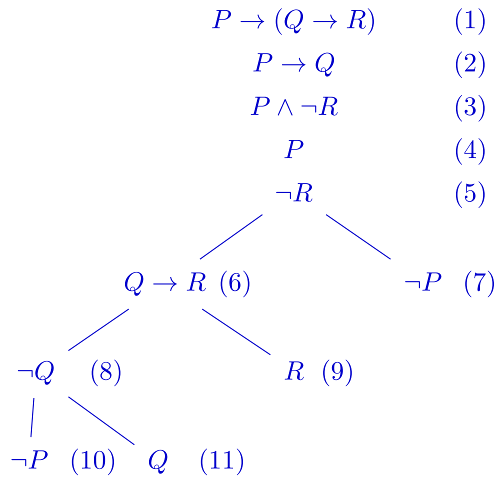
$$P \rightarrow Q \qquad (2)$$

$$P \wedge \neg R \qquad (3)$$

by exhibiting a strict tableau.

Proposed solution.

We construct a strict tableau for (1)–(3):



We start with α-expansion of (3), which yields (4) and (5). Then, β-expansion of (1) yields (6) and (7). Then, β-expansion of (6) yields (8) and (9). Finally, β-expansion of (2) yields (10) and (11).

Since both paths are now closed, the set of input formulas is unsatisfiable.

Exercise 8.2: Check whether the following propositional formulas are valid or not using semantic tableaux. Give a brief explanation. Use exactly the expansion rules given in the lecture.

(a) $(P \to Q) \to ((P \lor R) \to (Q \lor R))$

(b) $(P \lor Q) \to (P \land Q)$

Proposed solution. (a) We construct a strict tableau for the *negated* input formula (1):

$$\begin{array}{cr}
\neg\big((P \to Q) \to ((P \lor R) \to (Q \lor R))\big) & (1) \\
P \to Q & (2) \\
\neg((P \lor R) \to (Q \lor R)) & (3) \\
P \lor R & (4) \\
\neg(Q \lor R) & (5) \\
\neg Q & (6) \\
\neg R & (7)
\end{array}$$

$$\begin{array}{cc}
\neg P \quad (8) & Q \quad (9) \\
\begin{array}{cc} P \quad (10) & R \quad (11) \end{array} &
\end{array}$$

The α-expansion of (1) yields (2) and (3), α-expansion of (3) yields (4) and (5), α-expansion of (5) yields (6) and (7), β-expansion of (2) yields (8) and (9), and β-expansion of (4) yields (10) and (11). Since every path is now closed, the negated input formula is unsatisfiable, so the input formula is valid.

(b) We construct a strict tableau for the *negated* input formula:

$$
\begin{array}{ccccc}
 & & \neg((P \vee Q) \rightarrow (P \wedge Q)) \quad (1) & & \\
 & & P \vee Q \quad (2) & & \\
 & & \neg(P \wedge Q) \quad (3) & & \\
 & P \quad (4) & & Q \quad (5) & \\
\neg P \quad (6) & \neg Q \quad (7) & & \neg P \quad (8) & \neg Q \quad (9)
\end{array}
$$

The α-expansion of (1) yields (2) and (3), β-expansion of (2) yields (4) and (5), β-expansion of (3) yields (6) and (7), and once more β-expansion of (3) yields (8) and (9). Since the second (and also the third) path is now *maximal and open*, the set of formulas on this path is satisfiable. In particular, the negated input formula is satisfiable, so the input formula is not valid.

Exercise 8.3: Check whether the following propositional formulas are valid or not using semantic tableaux. Give a brief explanation. Use exactly the expansion rules given in the lecture.

(a) $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$.

(b) $(R \wedge (R \rightarrow P)) \rightarrow (P \wedge \neg Q)$.

Proposed solution. (a) We construct a tableau for the negation of the formula:

$$
\begin{array}{ccc}
 & \neg((P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))) & \\
 & P \rightarrow Q & \\
 & \neg((Q \rightarrow R) \rightarrow (P \rightarrow R)) & \\
 & Q \rightarrow R & \\
 & \neg(P \rightarrow R) & \\
 & P & \\
 & \neg R & \\
\neg P & & Q \\
 & \neg Q & R
\end{array}
$$

Since all paths are closed, the tableau is closed, so the original formula is valid.

(b) We construct a tableau for the negation of the formula:

$$
\begin{array}{c}
\neg((R \land (R \to P)) \to (P \land \neg Q)) \\
R \land (R \to P) \\
\neg(P \land \neg Q) \\
R \\
R \to P \\
\begin{array}{cc}
\diagup & \diagdown \\
\neg R & P \\
 & \begin{array}{cc} \diagup & \diagdown \\ \neg P & \neg\neg Q \\ & Q \end{array}
\end{array}
\end{array}
$$

Since the tableau has a maximal and open path, the negation of the original formula is satisfiable, so the original formula is not valid.

Exercise 8.4: Determine the satisfiability of the following set of ground formulas using the tableau calculus:

$$P(b) \land \neg P(d) \tag{1}$$

$$P(c) \lor \big(P(b) \land P(d)\big) \tag{2}$$

$$P(c) \to \neg\big(P(b) \lor P(d)\big) \tag{3}$$

Use exactly the expansion rules given in the lecture. State explicitly whether the set is satisfiable and give an explanation for that statement.

Proposed solution. We construct a strict tableau for (1)–(3):

$$
\begin{array}{c}
\begin{array}{cl}
P(b) \land \neg P(d) & (1) \\
P(c) \lor \big(P(b) \land P(d)\big) & (2) \\
P(c) \to \neg\big(P(b) \lor P(d)\big) & (3) \\
P(b) & (4) \\
\neg P(d) & (5)
\end{array} \\
\begin{array}{cc}
\diagup & \diagdown \\
\begin{array}{c}
P(c) \quad (6) \\
\begin{array}{cc}
\diagup & \diagdown \\
\neg P(c) \quad (10) & \begin{array}{cl} \neg\big(P(b) \lor P(d)\big) & (11) \\ \neg P(b) & (12) \\ \neg P(d) & (13) \end{array}
\end{array}
\end{array}
&
\begin{array}{cl}
P(b) \land P(d) & (7) \\
P(b) & (8) \\
P(d) & (9)
\end{array}
\end{array}
\end{array}
$$

We start with α-expansion of (1), this yields (4) and (5), then β-expansion of (2) yields (6) and (7), and α-expansion of (7) yields (8) and (9). The rightmost branch is now closed.

We continue with β-expansion of (3), this yields (10) and (11). The leftmost branch is now also closed.

Finally, α-expansion of (11) yields (12) and (13), so that the middle branch is closed as well.

Since every path is now closed, the set of input formulas is unsatisfiable.

Exercise 8.5: Extend the tableau calculus to support the following connectives:

- The Sheffer stroke, denoted $|$, is a binary connective meaning "not both." Thus, $F \,|\, G$ is equivalent to $\neg F \vee \neg G$.
- The Peirce arrow, denoted $\downarrow$, is a binary connective meaning "neither nor." Thus, $F \downarrow G$ is equivalent to $\neg F \wedge \neg G$.

Proposed solution.

Conjunctive			Disjunctive		
α	α_1	α_2	β	β_1	β_2
$F \wedge G$	F	G	$\neg(F \wedge G)$	$\neg F$	$\neg G$
$\neg(F \vee G)$	$\neg F$	$\neg G$	$F \vee G$	F	G
$\neg(F \rightarrow G)$	F	$\neg G$	$F \rightarrow G$	$\neg F$	G
$\neg(F \,\|\, G)$	F	G	$F \,\|\, G$	$\neg F$	$\neg G$
$F \downarrow G$	$\neg F$	$\neg G$	$\neg(F \downarrow G)$	F	G

Exercise 8.6: Refute the following set of formulas using the tableau calculus with ground instantiation:

$$\forall x\, \exists y\, P(x, y) \qquad (1)$$

$$\exists z\, \forall w\, \neg P(f(z), w) \qquad (2)$$

Proposed solution.

$$
\begin{array}{cr}
\forall x\, \exists y\, P(x,y) & (1) \\
\exists z\, \forall w\, \neg P(f(z),w) & (2) \\
\forall w\, \neg P(f(c),w) & (3) \\
\exists y\, P(f(c),y) & (4) \\
P(f(c),b) & (5) \\
\neg P(f(c),b) & (6)
\end{array}
$$

We start with δ-expansion of (2), skolemize and replace z with the Skolem constant c. This yields (3). Then γ-expansion of (1) yields (4), as we replace the universally quantified variable x with the ground term $f(c)$. We continue with δ-expansion of (4), skolemize and replace y with the Skolem constant b. This yields (5).

Finally, γ-expansion of (3) yields (6), as we replace the universally quantified variable w with the ground term b.

Since the path is now closed, the set of input formulas is unsatisfiable.

Exercise 8.7: Refute the following set of formulas using the free-variable tableau calculus:

$$
\begin{array}{cr}
\forall x\, \exists y\, P(x,y) & (1) \\
\exists z\, \forall w\, \neg P(f(z),w) & (2)
\end{array}
$$

Proposed solution.

$$
\begin{array}{cr}
\forall x\, \exists y\, P(x,y) & (1) \\
\exists z\, \forall w\, \neg P(f(z),w) & (2) \\
\exists y\, P(v_1,y) & (3) \\
\forall w\, \neg P(f(c),w) & (4) \\
P(v_1,b) & (5) \\
\neg P(f(c),v_2) & (6)
\end{array}
$$

We start with γ-expansion of (1) and replace the universally quantified variable x with the free variable v_1. This yields (3). Then δ-expansion of (2) yields (4), as we skolemize and replace z with the Skolem constant c. We continue with δ-expansion of (3), skolemize and replace y with the Skolem constant b. This yields (5).

Finally, γ-expansion of (4) yields (6), as we replace the universally quantified variable w with the free variable v_2.

5. and 6. are complementary (modulo unification):

$$v_1 \doteq f(c) \text{ and } v_2 \doteq b$$

Since the path is now closed, the set of input formulas is unsatisfiable.