Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

Winter Term 2024/25

Exercises 4: First-Order Logic

Exercise 4.1: Let $\Sigma = (\{b/0,\ c/0,\ d/0,\ f/1\}, \{P/1\})$. Does the formula

$$P(b) \land P(c) \land \neg P(d) \land \neg(\exists x\, P(f(f(x))))$$

have a Σ-model whose universe has exactly two elements? Give an example of such a model or show that such a model does not exist.

Exercise 4.2: Let the signature $\Sigma = (\Omega, \Pi)$ be given by $\Omega = \{+/2, s/1, 0/0\}$ and $\Pi = \emptyset$, and let

$$\begin{aligned} F_1 &= \forall x\, (x + 0 \approx x) \\ F_2 &= \forall x \forall y\, (x + s(y) \approx s(x + y)) \\ F_3 &= \forall x \forall y\, (x + y \approx y + x) \\ F_4 &= \neg \forall x \forall y\, (x + y \approx y + x). \end{aligned}$$

(a) Determine a Σ-algebra $\mathcal{A}$ with an universe of exactly two elements such that $\mathcal{A}$ is a model of F_1, F_2, F_3.

(b) Determine a Σ-algebra $\mathcal{A}$ with an universe of exactly two elements such that $\mathcal{A}$ is a model of F_1, F_2, F_4.

Exercise 4.3: Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1,\ c/0\}$. Give a Σ-model $\mathcal{A}$ of

$$\neg f(c) \approx c \quad \land \quad \forall x\, (f(f(x)) \approx x)$$

with $U_{\mathcal{A}} = \{1, 2, 3\}$.

Exercise 4.4: Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{f/2,\ g/2\}$ and $\Pi = \{P/2,\ Q/1\}$. Let

$$F \;=\; \forall x\, (P(x, y) \lor \exists y\, P(x, f(y, z)))$$

and $\sigma = \{y \mapsto g(x, z),\ z \mapsto g(x, y)\}$. Compute $F\sigma$.

Exercise 4.5: Let F be a formula. Prove that $\exists x\, F$ is satisfiable if and only if $F\{x \mapsto b\}$ is satisfiable, where b is a constant that does not occur in F.

Exercise 4.6: Let $\Sigma = (\Omega, \Pi)$ be a signature. For every Σ-formula F without equality, let $neg(F)$ be the formula that one obtains from F by replacing every atom $P(t_1, \dots, t_n)$ in F by its negation $\neg P(t_1, \dots, t_n)$ for every $P/n \in \Pi$. Prove: If F is valid, then $neg(F)$ is valid.

Hint: Somewhere in the proof you need an induction over the structure of formulas. It is sufficient if you check the base cases and $\land$, $\neg$, and $\exists$. The other boolean connectives and quantifiers ($\lor$, $\rightarrow$, $\leftrightarrow$, $\forall$) can be handled analogously; you may omit them.

Exercise 4.7 ($*$)**:** Let Π be a set of propositional variables. Let N and N' be sets of clauses over Π. Let S be a set of literals that does not contain any complementary literals. Prove: If every clause in N contains at least one literal L with $L \in S$ and if no clause in N' contains a literal L with $\overline{L} \in S$, then $N \cup N'$ is satisfiable if and only if N' is satisfiable.

Exercise 4.8: Let $\Sigma = (\Omega, \Pi)$ be a signature where Π contains two predicate symbols Q and R with the same arity n and possibly further predicate symbols. For any Σ-formula F let $rep(F)$ be the formula that one obtains by replacing every atom $Q(s_1, \dots, s_n)$ in F by the corresponding atom $R(s_1, \dots, s_n)$.

(a) Prove: If F is valid, then $rep(F)$ is valid. It is sufficient if you consider nonequational atoms, disjunctions $G \lor G'$, and negations $\neg G$; the other cases are handled analogously.

(b) Refute: If F is satisfiable, then $rep(F)$ is satisfiable.

Exercise 4.9 ($*$)**:** Let $\Sigma = (\Omega, \Pi)$ be a signature. Let $P/1$ and $Q/0$ be predicate symbols in Π. Let N be a set of (universally quantified) clauses over Σ. Let N_0 be the set of all clauses in N that contain a literal $\neg P(t)$ for some $t \in \mathrm{T}_\Sigma(X)$. Let N_1 be the set of all clauses in N that contain a literal $P(t')$ for some $t' \in \mathrm{T}_\Sigma(X)$. Prove: If all clauses in $N_0 \setminus N_1$ contain also the literal $\neg Q$ and if all clauses in $N_1 \setminus N_0$ contain also the literal Q, then N and $(N \setminus N_0) \setminus N_1$ are equisatisfiable.

Exercise 4.10 (*)**:** Let $\succ$ be a well-founded strict partial ordering on a set M. A function $\phi : M^n \to M$ with $n \geq 1$ is called strictly monotonic in the jth argument if $a_j \succ a'_j$ implies $\phi(a_1, \dots, a_j, \dots, a_n) \succ \phi(a_1, \dots, a'_j, \dots, a_n)$ for all arguments $a_1, \dots,$ $a_n, a'_j \in M$.

(a) Prove: If the ordering $\succ$ on the set M is well-founded and total, and if $\phi : M^n \to M$ with $n \geq 1$ is strictly monotonic in the jth argument, then $\phi(a_1, \dots, a_j, \dots, a_n) \succeq a_j$ for all $a_1, \dots, a_n \in M$.

(b) In part (a), it was required that $\succ$ is a *total* ordering. Give an example that shows that the property of part (a) does not hold if the ordering $\succ$ is well-founded but not total.

(c) Use part (a) to prove the following property: Let $\Sigma = (\Omega, \Pi)$ be a signature, let $\mathcal{A}$ be a Σ-algebra. Let $\succ$ be a well-founded total ordering on the universe $U_{\mathcal{A}}$ of $\mathcal{A}$, such that $f_{\mathcal{A}} : U_{\mathcal{A}}^n \to U_{\mathcal{A}}$ is strictly monotonic in every argument for every $f/n \in \Omega$ with $n \geq 1$. Let β be an arbitrary $\mathcal{A}$-assignment, let $t \in \mathrm{T}_\Sigma(X)$. Then $\mathcal{A}(\beta)(t) \succeq \beta(x)$ for every variable $x \in \mathrm{var}(t)$.