Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 3: Propositional Logic Continued

Exercise 3.1: A partial Π-valuation $\mathcal{A}$ under which all clauses of a clause set N are true is called a partial Π-model of N.

Do the following clause sets over $\Pi = \{P, Q, R\}$ have partial Π-models that are not total Π-models (that is, models in the sense of Sect. 2.3)? If yes, give such a partial Π-model.

(1)
$$\begin{array}{rcrcr} P & & & & \\ \neg P & \vee & Q & & \\ \neg P & \vee & \neg Q & \vee & \neg R \end{array}$$

(2)
$$\begin{array}{rcrcr} P & & & & \\ \neg P & \vee & Q & & \\ & & \neg Q & \vee & R \\ \neg P & \vee & \neg Q & \vee & \neg R \end{array}$$

(3)
$$\begin{array}{rcrcr} P & & & \vee & R \\ \neg P & \vee & Q & \vee & \neg R \\ & & \neg Q & \vee & \neg R \end{array}$$

(4)
$$\begin{array}{rcrcr} \neg P & \vee & Q & & \\ & & \neg Q & \vee & R \\ P & & & \vee & \neg R \end{array}$$

Exercise 3.2: For any propositional formula F, let $\mathit{negvar}(F)$ be the formula obtained from F by replacing every propositional variable by its negation. Formally:

$$\begin{aligned} negvar(P) &= \neg P \\ negvar(\neg G) &= \neg negvar(G) \\ negvar(G_1 \wedge G_2) &= negvar(G_1) \wedge negvar(G_2) \end{aligned}$$

and so on. For example, $negvar(P \vee (\neg Q \rightarrow (\neg P \wedge \top))) = \neg P \vee (\neg\neg Q \rightarrow (\neg\neg P \wedge \top))$.

Prove or refute: If a formula F is satisfiable, then $negvar(F)$ is satisfiable. (It is sufficient if you consider the boolean connectives $\neg$ and $\wedge$; the others are treated analogously.)

Exercise 3.3: Let N be the following set of propositional clauses over $\Pi = \{P, Q, R\}$:

$$\begin{array}{ccccccc} P & \vee & \neg Q & & & \qquad & (1) \\ & & Q & \vee & \neg R & \qquad & (2) \\ \neg P & & & \vee & R & \qquad & (3) \end{array}$$

(a) Use the DPLL procedure to compute a (total) model of N.

(b) Use the DPLL procedure to prove that $N \models R \rightarrow P$. Before you can invoke the procedure, you will first need to transform the entailment into a suitable set of clauses.

Exercise 3.4 (∗)**:** A friend asks you to proofread her bachelor thesis. On page 14 of the thesis, she writes the following:

> **Definition 11.** Let N be a set of propositional formulas. The set $poscomb(N)$ of positive combinations of formulas in N is defined inductively by
> (1) $N \subseteq poscomb(N)$;
> (2) if $F, F' \in poscomb(N)$, then $F \vee F' \in poscomb(N)$; and
> (3) if $F, F' \in poscomb(N)$, then $F \wedge F' \in poscomb(N)$.
>
> **Lemma 12.** If N is a satisfiable set of formulas, then every positive combination of formulas in N is satisfiable.
>
> **Proof.** The proof proceeds by induction over the formula structure. Let $G \in poscomb(N)$. If $G \in N$, then it is obviously satisfiable, since N is satisfiable. Otherwise, G must be a disjunction or a conjunction of formulas in $poscomb(N)$. If G is a disjunction $F \vee F'$ with $F, F' \in poscomb(N)$, we know by the induction hypothesis that F is satisfiable. So F has a model. Since this is also a model of $G = F \vee F'$, the formula G is satisfiable. Analogously, if G is a conjunction $F \wedge F'$, with $F, F' \in poscomb(N)$, then both F and F' are satisfiable by induction, so $G = F \wedge F'$ is satisfiable as well.

(1) Is the "proof" correct?

(2) If the "proof" is not correct:

   (a) Which step is incorrect?

   (b) Does the "lemma" hold? If yes, give a correct proof; otherwise, give a counterexample.

Exercise 3.5: The sudoku puzzle presented in the first lecture has a unique solution.

	1	2	3	4	5	6	7	8	9
1								1	
2	**4**								
3		2							
4					5		4		7
5			8				3		
6			1		9				
7	3			4			2		
8		5		1					
9				8		6			

If we replace the 4 in column 1, row 2 by some other digit, this need no longer hold. Use a SAT solver to find out for which values in column 1, row 2 the puzzle has no solution.

Hint: The Perl script at

`https://rg1-teaching.mpi-inf.mpg.de/autrea-ws23/gensud`

produces an encoding of the sudoku above in DIMACS CNF format, which is accepted by most SAT solvers.

Exercise 3.6 ($*$)**:** Given a sudoku puzzle, briefly describe a set of propositional clauses that is satisfiable if and only if the puzzle has more than one solution.

Exercise 3.7: A finite graph is a pair (V, E), where V is a finite nonempty set and $E \subseteq V \times V$. The elements of V are called vertices or nodes; the elements of E are called edges. A graph has a 3-coloring if there exists a function $\phi : V \to \{0, 1, 2\}$ such that for every edge $(v, v') \in E$ we have $\phi(v) \neq \phi(v')$.

Give a linear-time translation from finite graphs (V, E) to propositional clause sets N such that (V, E) has a 3-coloring if and only if N is satisfiable and such that every model of N corresponds to a 3-coloring ϕ and vice versa.

Exercise 3.8 $(*)$**:** A finite graph is a pair (V, E), where V is a finite nonempty set and $E \subseteq V \times V$. The elements of V are called vertices or nodes; the elements of E are called edges. A graph has a 3-coloring if there exists a function $\phi : V \to \{0, 1, 2\}$ such that for every edge $(v, v') \in E$ we have $\phi(v) \neq \phi(v')$. A 3-coloring is called complete if for every pair $(c, c') \in \{0, 1, 2\} \times \{0, 1, 2\}$ with $c \neq c'$ there exists an edge $(v, v') \in E$ such that $\phi(v) = c$ and $\phi(v') = c'$ or $\phi(v) = c'$ and $\phi(v') = c$.

Give a linear-time translation from finite graphs (V, E) to propositional clause sets N such that (V, E) has a complete 3-coloring if and only if N is satisfiable and such that every model of N corresponds to a complete 3-coloring ϕ and vice versa.

Exercise 3.9: Give OBDDs for the following three formulas:

(a) $\neg P$

(b) $P \leftrightarrow Q$

(c) $(P \land Q) \lor (Q \land R) \lor (R \land P)$

Consider the ordering $P < Q < R$.

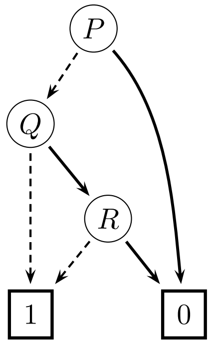
Exercise 3.10: Let F be the propositional formula $P \land (Q \lor R) \land S$.

(a) Give the reduced OBDD for F w.r.t. the ordering $P < Q < R < S$.

(b) Find a total ordering over $\{P, Q, R, S\}$ such that the reduced OBDD for F has 6 nonleaf nodes. Give the resulting reduced OBDD.

(c) For how many total orderings over $\{P, Q, R, S\}$ does the reduced OBDD for F have 6 nonleaf nodes?

Exercise 3.11: (a) Give a propositional formula F that is represented by this reduced OBDD:



(b) How many different reduced OBDDs over the propositional variables $\{P, Q, R\}$ have exactly one interior (nonleaf) node?

(c) Find a propositional formula G over the propositional variables $\{P, Q, R\}$, such that the reduced OBDD for G has three interior nodes and the reduced OBDD for $F \vee G$ has one interior node. Give the reduced OBDDs for G and $F \vee G$.

Exercise 3.12 ($*$)**:** Let F_n be a propositional formula over $\{P_1, \ldots, P_n\}$ such that $\mathcal{A}(F_n) = 1$ if and only if $\mathcal{A}$ maps exactly one of the propositional variables $P_1, \ldots, P_n$ to 1 and the others to 0. How many nodes does a reduced OBDD for F_n have (including the leaf nodes $\boxed{0}$ and $\boxed{1}$)?