Automated Theorem Proving

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based on exercises by Dr. Uwe Waldmann

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Exercises 2: Preliminaries Continued and Propositional Logic

Exercise 2.1: Determine all strict total orderings $\succ$ on the set $\{a, b, c, d, e\}$ such that the following properties hold simultaneously:

(1) $\{a, b\} \succ_{\text{mul}} \{a, a, c\}$

(2) $\{c, d\} \succ_{\text{mul}} \{b, b, b\}$

(3) $\{a, e\} \succ_{\text{mul}} \{c, e, e\}$

Exercise 2.2: Let M be a set, and let $\succ$ be a strict partial ordering over M. Let $b, b_1, b_2 \in M$, and let S, S_1, S_2 be finite multisets over M.

(a) Prove or refute: If $\{b\} \succ_{\text{mul}} S_1$ and $\{b\} \succ_{\text{mul}} S_2$, then $\{b\} \succ_{\text{mul}} S_1 \cup S_2$.

(b) Prove or refute: If $S \succ_{\text{mul}} \{b_1\}$ and $S \succ_{\text{mul}} \{b_2\}$, then $S \succ_{\text{mul}} \{b_1, b_2\}$.

Exercise 2.3: (a) Let $M = \{a, b, c, d\}$. Suppose that the binary relation $\to$ over multisets over M is defined by the rules (1)–(3):

(1) $\quad S \cup \{b, c\} \to S \cup \{a, a, a\}$
(2) $\quad S \cup \{b, a\} \to S \cup \{b, c, c\}$
(3) $\quad S \cup \{c\} \to S \cup \{d\}$

Then $\to$ can be shown to be terminating using the multiset extension $\succ_{\text{mul}}$ of an appropriate well-founded ordering on M. What does $\succ$ look like?

(b) If the binary relation $\to$ is defined by the rules (4)–(6),

(4)	$S \cup \{a, a\}$	$\to$	$S \cup \{b, c\}$
(5)	$S \cup \{b, b\}$	$\to$	$S \cup \{a, c\}$
(6)	$S \cup \{b, c\}$	$\to$	$S \cup \{a, d, c, c\}$

then there is no well-founded ordering on M such that $\to$ is contained in $\succ_{\text{mul}}$. Why? Give a short explanation.

(c) Nevertheless, the relation $\to$ defined by the rules (4)–(6) is terminating. Prove it. (Hint: Think about lexicographic combinations.)

Exercise 2.4 (∗)**:** Prove: If S and S' are finite multisets over a set M, and $S \succ_{\text{mul}} S'$ holds for *every* strict partial ordering $\succ$ over M, then $S' \subset S$ (that is, $S' \subseteq S$ and $S' \neq S$).

Exercise 2.5: Which of the following propositional formulas are valid? Which are satisfiable?

(1) $\neg P$

(2) $P \to \bot$

(3) $\bot \to P$

(4) $(P \vee Q) \to P$

(5) $P \to (Q \to P)$

(6) $Q \to \neg Q$

(7) $Q \wedge \neg Q$

(8) $\neg(\neg P \wedge \neg\neg P)$

Exercise 2.6 (∗)**:** Let $N = \{C_1, \ldots, C_n\}$ be a finite set of propositional clauses without duplicated literals or complementary literals such that for every $i \in \{1, \ldots, n\}$ the clause C_i has exactly i literals. Prove or refute: N is satisfiable.

Exercise 2.7: Let F, G, H be propositional formulas, let p be a position of H. Prove or refute: If $H[F]_p$ is valid and $H[G]_p$ is valid, then $H[F \vee G]_p$ is valid.

Exercise 2.8: Let F, G, H be propositional formulas, let p be a position of H. Prove or refute: If $H[F \wedge G]_p$ is valid, then $H[F]_p$ and $H[G]_p$ are valid.

Exercise 2.9: Let Π be a set of propositional variables with $P, Q \in \Pi$. For every propositional formula F over Π, let $\phi(F)$ be the formula that one obtains from F by replacing every occurrence of P by $P \vee Q$. For instance, if $F = ((R \vee \neg P) \wedge (Q \vee P))$, then $\phi(F) = ((R \vee \neg(P \vee Q)) \wedge (Q \vee (P \vee Q)))$, and if $F = R$, then $\phi(F) = R$.

(a) Prove: If $\phi(F)$ is satisfiable, then F is satisfiable. (Note: It is sufficient if you consider propositional variables, negations, and conjunctions; the other cases are treated analogously.)

(b) Refute: If $\phi(F)$ is valid, then F is valid.

Exercise 2.10: Let Π be a set of propositional variables. Let Q and R be two propositional variables in Π. For any Π-formula F let $\phi(F)$ be the formula that one obtains by replacing every occurrence of Q in F by R.

Prove: If $\phi(F)$ is satisfiable, then F is satisfiable. (It is sufficient if you consider propositional variables, conjunctions, and negations; the other cases are handled analogously.)

Exercise 2.11: Let N be a set of propositional clauses. Prove or refute the following statement: If N contains clauses $C_i \vee D_i$ ($i \in \{1, \ldots, n\}$) such that $\{C_i \mid i \in \{1, \ldots, n\}\} \models \bot$, then $N \models \bigvee_{i \in \{1, \ldots, n\}} D_i$.